(1) Prove that $\frac{\sqrt{2} + \sqrt{5}}{6}$ is not constructible.

**Solution**

Suppose $x_0 = \frac{\sqrt{2} + \sqrt{5}}{6}$ is constructible. Since $\sqrt{2}$ and 6 are constructible this implies that $6x_0 - \sqrt{2} = \frac{\sqrt{5}}{3}$ is constructible.

$\sqrt{5}$ is a solution of the equation $x^3 - 5 = 0$. By a theorem from class if a cubic equation with rational coefficients has a constructible solution it also has a rational one. Therefore $x^3 - 5 = 0$ must have a rational solution. We can write it in the form $\frac{m}{n}$ where $m$ and $n$ are relatively prime. Then by another theorem from class we must have that $m|5$ and $n|1$. Therefore $\frac{m}{n} = \pm 1$ or $\pm 5$. Plugging those numbers into $x^3 - 5 = 0$ we see that none of them are solutions. This is a contradiction and therefore $x_0$ is not constructible.

(2) Prove that $\frac{\pi^2}{3}$ is not constructible.

**Solution**

Suppose $x_0 = \frac{\pi^2}{3}$ is constructible. Then $3x_0 = \pi^2$ is also constructible and hence $\pi = \sqrt{3x_0}$ is constructible too. However, every constructible number is algebraic and $\pi$ is not algebraic. This is a contradiction and hence $\frac{\pi^2}{3}$ is not constructible.

(3) Let $F$ be the field consisting of real numbers of the form $p+q\sqrt{2} + \sqrt{3}$ where $p, q$ are of the form $a + b\sqrt{2}$, with $a, b$ rational. Represent $\frac{1 + \sqrt{2} + \sqrt{3}}{2 - 3\sqrt{2} + \sqrt{3}}$ in this form.

**Solution**

$$\frac{1 + \sqrt{2} + \sqrt{3}}{2 - 3\sqrt{2} + \sqrt{3}} = \frac{(1 + \sqrt{2} + \sqrt{3})(2 + 3\sqrt{2} + \sqrt{3})}{(2 - 3\sqrt{2} + \sqrt{3})(2 + 3\sqrt{2} + \sqrt{3})}$$

$$= \frac{2 + 3(2 + \sqrt{2}) + 5\sqrt{2} + \sqrt{3}}{4 - 9(2 + \sqrt{2})} = \frac{8 + 3\sqrt{2} + 5\sqrt{2} + \sqrt{3}}{-14 - 9\sqrt{2}}$$

$$= \frac{(8 + 3\sqrt{2} + 5\sqrt{2} + \sqrt{3})(-14 + 9\sqrt{2})}{(-14 - 9\sqrt{2})(-14 + 9\sqrt{2})}$$

$$= \frac{-112 - 52\sqrt{2} - 70\sqrt{2} + \sqrt{3} + 126 + 45\sqrt{2} + \sqrt{2} + \sqrt{2}}{196 - 2 \cdot 81}$$

$$= \frac{-58 + 20\sqrt{2} + (-70 + 45\sqrt{2})\sqrt{2} + \sqrt{2}}{4}$$

(4) Find a tower of fields $Q = F_0 \subset F_1 \subset F_2 \subset F_3$ such that $\sqrt[3]{1 + \sqrt{2} + \sqrt{3}} \in F_3$. 


Show that all the steps in the tower except for the last one are nontrivial. I.e show that $F_0 \neq F_1$, and $F_1 \neq F_2$.

Solution

Let $F_0 = \mathbb{Q}$, $F_1 = F_0(\sqrt{2})$, $F_2 = F_1(\sqrt{3})$, $F_3 = F_2(\sqrt{1 + \sqrt{2} + \sqrt{3}})$.

Note that $F_1 \neq F_0$ since $\sqrt{2}$ is irrational. To see that $F_2 \neq F_1$ suppose $F_2 = F_1$. Then $\sqrt{3} \in F_1$ and we can write $\sqrt{3} = a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$. It’s easy to see that we must have $a \neq 0, b \neq 0$. Squaring both sides of the above formula, we get $3 = (a + b\sqrt{2})^2 = a^2 + 2b^2 + 2ab\sqrt{2}$. This implies that $\sqrt{2} = \frac{3 - a^2 - 2b^2}{2ab}$ is rational. This is a contradiction and therefore $F_2 \neq F_1$.

(5) Show that none of the real roots of $3x^3 - 2x^2 - 2 = 0$ are constructible.

Solution

This is a cubic equation with rational coefficients. By a theorem from class if this equation has a constructible root it also has a rational one. Let $x_0 = \frac{m}{n}$ be a rational root where $(m, n) = 1$. Then $m|2$ and $n|3$. Therefore $\frac{m}{n} = \pm 1, \pm 2, \pm \frac{2}{3}, \pm \frac{1}{3}$. Plugging in those numbers into the equation we see that none of them are roots.

(6) Let $0 < \theta < \pi/2$ be the angle with $\cos \theta = \frac{2}{7}$. Show that the angle $\theta$ is constructible but $\theta/3$ is not.

Solution

$\cos \theta = \frac{2}{7}$ is rational which means that the angle $\theta$ is constructible. Let $x = \cos(\theta/3)$. Suppose the angle $\theta/3$ is constructible. Then $x_0$ is a constructible number. By the formula $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$ we have that $x_0$ is a root of $4x^3 - 3x = \frac{2}{7}, 28x^3 - 14x - 2 = 0$.

By the theorem mentioned above if this equation has a constructible root it must also have a rational root. Let $x_1 = \frac{m}{n}$ be a rational root where $(m, n) = 1$. Then $m|28$ and $n|2$. Therefore $\frac{m}{n} = \pm 1, \pm 2, \pm \frac{1}{7}, \pm \frac{1}{14}, \pm \frac{1}{28}$. Plugging in those numbers into the equation we see that none of them are roots.

(7) Show that the equation $x^9 - 4x^3 + 1 = 0$ has no constructible roots.

Solution

Suppose $x_0$ is a constructible root. Then $y_0 = x_0^3$ is also constructible. It satisfies $y_3 - 4y + 1 = 0$. As before, this is a cubic equation with rational coefficients and if it has a constructible root then it also has a rational one. Let $y_1 = \frac{m}{n}$ be a rational root where $(m, n) = 1$. Then $m|1$ and $n|1$. Therefore $\frac{m}{n} = \pm 1$. Plugging in $y = \pm 1$ into $y^3 - 4y + 1 = 0$ we see that none of them are roots.