1. Examples of smooth manifolds

Recall from the previous lecture that if $c$ is a regular value of $F: U \to \mathbb{R}^k$ where $U$ is an open subset of $\mathbb{R}^{n+k}$ then $M = \{F = c\}$ has a natural structure of a generalized manifold of dimension $n$.

**Example 1.0.1.** Let $SL(n, \mathbb{R})$ be the set of all $n \times n$ matrices with determinant 1. It is a smooth manifold of dimension $n^2 - 1$. To see this let $M(n \times m)$ be the set of all real $n \times m$ matrices. It can be canonically identified with $\mathbb{R}^{nm}$. $SL(n, \mathbb{R})$ is a subset of $M(n \times n)$ equal to the level set $\{F = 1\}$ of the function $F: M(n \times n) \to \mathbb{R}$ given by $F(A) = \det A$. Since the formula for the determinant of a matrix is polynomial in the coefficients of the matrix, it is obviously smooth.

We claim that 1 is a regular value of $F$. To see this we need to show that the differential $dF_A: \mathbb{R}^{n^2} \to \mathbb{R}$ is onto for any $A \in SL(n, \mathbb{R})$. Since the target is 1-dimensional this is equivalent to checking that $dF_A \neq 0$.

Let’s find the formula for $dF_A$. We first consider the case $A = Id$.

**Claim:**

$$dF_{Id}(X) = \text{tr}(X)$$

for any $X \in M(n \times n)$.

Since both the left and the right side of this formula are linear in $X$ it’s enough to verify it on the standard basis of $M(n \times n) = \mathbb{R}^{n^2}$.

Let $E_{ij}$ be the $n \times n$ matrix which has the $(i, j)$ entry equal to 1 and all other entries equal to 0.

Suppose $i \neq j$. Then $\text{tr}(E_{ij}) = 0$. On the other hand, $dF_{Id}(E_{ij}) = D_{E_{ij}} F_{Id} = \lim_{t \to 0} \frac{F(Id + tE_{ij}) - F(Id)}{t} = \lim_{t \to 0} \frac{1}{t} 1 = 0$ since $Id + tE_{ij}$ is a triangular matrix with 1’s on the diagonal.

Thus $dF_{Id}(E_{ij}) = \text{tr}(E_{ij})$ for any $i \neq j$.

Let’s now consider the case $i = j$. Obviously, $\text{tr}(E_{ii}) = 1$.

As before we compute $dF_{Id}(E_{ii}) = D_{E_{ii}} F_{Id} = \lim_{t \to 0} \frac{F(Id + tE_{ii}) - F(Id)}{t} = \lim_{t \to 0} \frac{1 + \frac{t}{t} 1}{t} = 1$ because $Id + tE_{ii}$ is a diagonal matrix with the $i$th diagonal entry equal to 1 and $t$ and the rest of of diagonal elements equal to 1. $dF_{Id}(E_{ii}) = \text{tr}(E_{ii})$ for any $i$. Together with the above this means that $dF_{Id}(E_{ij}) = \text{tr}(E_{ij})$ for any $i, j$. By linearity of $\text{tr}$ and $dF_{Id}$ this proves the **Claim**.

Now suppose $A$ is an arbitrary matrix in $SL(n, \mathbb{R})$ and let $X \subset M(n \times n)$ be any matrix.

Then, using multiplicativity of determinants and the Claim above we compute

$$dF_A(X) = \lim_{t \to 0} \frac{F(A + tX) - F(A)}{t} = \lim_{t \to 0} \frac{\det(A + tX) - \det(A)}{t} =$$

$$= \lim_{t \to 0} \frac{\det(A(Id + tA^{-1}X) - \det(A)}{t} = \lim_{t \to 0} \frac{\det(A) \det(Id + tA^{-1}X) - \det(A)}{t} =$$
Thus
\[
dF_A(X) = tr(A^{-1}X)
\]
This obviously means that \( dF_A \neq 0 \) (e.g. because \( dF_A(A) = tr(A^{-1}A) = n \)). Therefore \( 1 \) is a regular value of \( F \) and hence \( SL(n, \mathbb{R}) = \{ F = 1 \} \) is a smooth manifold of dimension \( n^2 - 1 \).

**Example 1.0.2.** Let \( O(n) = \{ A \in M(n \times n) \mid A \cdot A^t = Id \} \). Then \( O(n) \) is a smooth manifold. To see this, consider the map \( F: M(n \times n) \rightarrow M(n \times n) \) given by \( F(A) = A \cdot A^t \). Then \( O(n) = \{ F = Id \} \). However, \( Id \) is not a regular value because \( (A \cdot A^t)^t = (A^t)^t \cdot A^t = A \cdot A^t \) which means that \( (A \cdot A^t) \) is symmetric for any \( A \). Let \( Sym(n) \) be the set of symmetric \( n \times n \) matrices. Then \( F \) maps \( M(n \times n) \) to \( Sym(n) \).

**Claim.** \( Id \) is a regular value of \( F: M(n \times n) \rightarrow Sym(n) \) Consequently, \( O(n) \) is a smooth manifold of dimension \( n(n - 1)/2 \). (Homework).

2. Manifolds with boundary

Let \( \mathbb{H}^n = \mathbb{R}^n_+ = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0 \} \). We will call the set \( \{ x_n > 0 \} \) the interior of \( \mathbb{H}^n \) and denote it by \( \text{int} \mathbb{H}^n \). We will call the set \( \{ x_n = 0 \} \) the boundary of \( \mathbb{H}^n \) and denote it by \( \text{bd} \mathbb{H}^n \).

A subset \( U \subset H^n \) is said to be open in \( H^n \) if \( U = W \cap H^n \) for some open set \( W \subset \mathbb{R}^n \).

**Definition 2.0.3.** Let \( U \subset \mathbb{H}^n \) be open. A map \( F: U \rightarrow \mathbb{R}^k \) is called smooth if for every point \( x \in U \) there exists an open set \( W \subset \mathbb{R}^n \) containing \( x \) such that \( W \cap \mathbb{H}^n \subset U \) and \( F|_{U \cap W} \) admits a smooth extension \( \tilde{F}: W \rightarrow \mathbb{R}^k \).

(Note that \( \tilde{F} \) need not be unique).

**Definition 2.0.4.** A generalized smooth manifold with boundary is a set \( X \) together with a atlas \( \{ \psi_\alpha : V_\alpha \rightarrow U_\alpha \subset X \}_{\alpha \in A} \) where \( V_\alpha \) is an open subset of \( H^n \) such that the following properties are satisfied

1. \( \cup_\alpha U_\alpha = M \)
2. \( \psi_\alpha : V_\alpha \rightarrow U_\alpha \) is a bijection for every \( \alpha \)
3. For any \( \alpha, \beta \) the set \( U_{\alpha \beta} = \psi_\beta^{-1}(U_\alpha \cap U_\beta) \) is open in \( \mathbb{R}^n \)
4. For any \( \alpha, \beta \) the map \( \psi_\beta^{-1} \circ \varphi_\alpha : U_{\alpha \beta} \rightarrow U_{\beta \alpha} \) is smooth.

Similar to usual manifolds one can define equivalence of atlases and prove the existence of a unique maximal atlas containing a given atlas for (generalized) manifolds with boundary.

As for manifolds without boundary one can define the topology on a generalized manifold with boundary: A subset \( U \subset X \) is called open if \( U \cap \psi_\alpha^{-1}(U_\alpha) \) is open in \( \mathbb{H}^n \).
Definition 2.0.5. A generalized smooth manifold $X$ with boundary is called a smooth manifold with boundary if it admits a countable atlas and is Hausdorff.

Example 2.0.6.

- $\mathbb{H}^n$ is a smooth manifold with boundary
- $\overline{D}^n = \{ x \in \mathbb{R}^n \mid |x| \leq 1 \}$ is a smooth manifold with boundary
- A (generalized) smooth manifold is a (generalized) smooth manifold with boundary (why?) If $M^n$ is a smooth manifold of dimension $n$ and $N^m$ is a smooth manifold with boundary of dimension $m$ then $M \times N$ is a smooth manifold with boundary of dimension $n + m$.
- Let $U \subset \mathbb{R}^n$ be open and let $F: U \to \mathbb{R}$ be smooth. Suppose $c \in \mathbb{R}$ is a regular value of $f$. Then $\{ F \leq c \}$ and $\{ F \geq c \}$ are smooth manifolds with boundary of dimension $n$.

Proof. We will construct an atlas on $M = \{ F \geq c \}$ as follows. Let $F(p) > c$. Then because $F$ is continuous, there is an $\epsilon > 0$ such that $B_\epsilon(p) \subset \{ F > c \}$. Pick a sufficiently large $d > 0$ such that $B_d(p + (0, \ldots, 0, d) \subset \mathbb{H}^n$. Set $V_p = B_d(p + (0, \ldots, 0, d)$ and define $\psi_p: V_p \to M$ by the formula $\psi_p(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, x_n - d)$. Now suppose $F(p) = c$. Since $c$ is a regular value of $F$ we have that $\left( \frac{\partial F}{\partial x_1}(p), \ldots, \frac{\partial F}{\partial x_n}(p) \right) \neq 0$. For simplicity let’s assume $\frac{\partial F}{\partial x_n}(p) \neq 0$. As in the proof of the Implicit function theorem consider the map $\Phi: U \to \mathbb{R}^n$ defined by $\Phi(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, F(x_1, \ldots, x_{n-1}, x_n) - c)$. We have that the matrix of partial derivatives $[d\Phi(p)]$ of $\Phi$ is an upper triangular matrix with 1’s on the diagonal except for the last entry which is equal to $\frac{\partial F}{\partial x_n}(p)$. Therefore $\det[d\Phi(p)] = \frac{\partial F}{\partial x_n}(p) \neq 0$ and the Inverse Function Theorem is applicable. It says that for some small $\epsilon$ the map $\Phi$ bijectively maps $B_\epsilon(p)$ to an open set $W$ in $\mathbb{R}^n$ and its inverse $\Psi: W \to B_\epsilon(p)$ is also smooth. Note that by construction $\Phi(\{ F \geq c \} \cap B_\epsilon(p)) = W \cap \mathbb{H}^n$ and $\Phi(\{ F = c \} \cap B_\epsilon(p)) = W \cap \partial \mathbb{H}^n$. Let $\Psi_p = \Phi^{-1}: V_p = W \cap \mathbb{H}^n \to (\{ F \geq c \} \cap B_\epsilon(p)$. This will be our parameterization of $M$ near $p$.

It is now easy to see that the collection of all $\Psi_p$ over $p \in M$ gives a smooth atlas satisfying the definition of a generalized manifold with boundary.

The fact that the resulting generalized manifold with boundary is Hausdorff and admits a countable atlas is proved in the same way as for regular level sets and is left to the reader as an exercise.

Definition 2.0.7. Let $X$ be a (generalized) smooth manifold with boundary. A point $p \in X$ is called interior if for some chart $\psi_\alpha: V_\alpha \to U_\alpha \subset X$ we have that $p = \psi_\alpha(p_\alpha)$ for some $p_\alpha \in \text{int} \mathbb{H}^n$. 

\[ \square \]
A point $p$ is called a boundary point of $X$ if for some chart $\psi_\alpha: V_\alpha \to U_\alpha \subset X$ we have that $p = \psi_\alpha(p_\alpha)$ for some $p_\alpha \in \partial \mathbb{H}^n$.

**Theorem 2.0.8** (Topological invariance of the boundary). Let $X^n$ be a (generalized) smooth manifold with boundary of dimension $n$.

1. Every point $p$ of $X$ is either an interior point of $X$ or a boundary point of $X$ but not both.
2. Let $\text{int} X$ be the set of interior points of $X$ and let $\partial X$ be the set of boundary points of $X$. Then $\text{int} X$ is a a (generalized) smooth manifold of dimension $n$ and $\partial X$ is a a (generalized) smooth manifold of dimension $n - 1$.

**Proof.** To see (1) suppose that $p$ is both an interior and a boundary point of $X$ at the same time. This means that there exists parameterizations $\psi_\alpha: V_\alpha \to U_\alpha \ni p$ and $\psi_\beta: V_\beta \to U_\beta \ni p$ such that $V_\alpha, V_\beta$ are open subsets of $\mathbb{H}^n$, $p = \psi_\alpha(p_\alpha) = \psi_\beta(p_\beta)$ and $p_\alpha \in \text{int} \mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$ and $p_\beta \in \partial \mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n = 0\}$. By taking a small ball around $p_\alpha$ and reducing the domain of $\psi_\alpha$ we can assume that $V_\alpha$ is an open subset of $\mathbb{R}^n$.

Consider the transition maps $\psi_{\alpha\beta} = \psi_\beta^{-1} \circ \psi_\alpha: V_\alpha \rightarrow V_{\beta}$ and $\psi_{\beta\alpha}: V_{\beta} \rightarrow V_\alpha$.

They are inverse to each other and WLOG we can assume that $\psi_{\beta\alpha}$ admits a smooth extension $\tilde{\psi}_{\beta\alpha}$ to a set $W_{\beta\alpha}$ which is open in $\mathbb{R}^n$. Since $\tilde{\psi}_{\beta\alpha} \circ \psi_{\alpha\beta} = \psi_{\beta\alpha} \circ \psi_{\alpha\beta} = \text{Id}$ by the chain rule we conclude that $d_{p_\alpha} \psi_{\alpha\beta}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isomorphism.

By the Inverse Function Theorem this means that there exist an open ball $B_{\epsilon}(p_\alpha) \subset V_{\alpha\beta}$ such that its image under $\psi_{\alpha\beta}$ is an open subset of $\mathbb{R}^n$ containing $p_\beta$.

This is a contradiction because the image of $V_{\alpha\beta}$ under $\psi_{\alpha\beta}$ is equal to $V_{\beta\alpha}$ which is a subset of $\mathbb{H}^n$ and no ball around $p_\beta$ is contained in $\mathbb{H}^n$.

Part (2) follows easily from part (1) (Exercise) \qed

**Remark 2.0.9.** A smooth manifold with boundary $X$ is as smooth manifold (without boundary) if and only if $\partial X = \emptyset$. 
