Informally a smooth manifold is a space which locally looks like an open subset of \( \mathbb{R}^n \). It need not be a subset of \( \mathbb{R}^N \) globally.

**Example 0.0.1.** Let \( X \) be the configuration space of two distinct points in \( \mathbb{R}^2 \) i.e. the set of two point subsets of \( \mathbb{R}^2 \). It’s equal to \( (\mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta \mathbb{R}^2)/\sim \) where \((x, y) \sim (y, x)\). This space is a manifold but it’s not canonically embedded into any \( \mathbb{R}^N \).

**Definition 0.0.2.** A generalized smooth manifold of dimension \( n \) is a set \( M \) together with a collection of subsets \{\( U_\alpha \)\}_{\alpha \in \mathcal{A}}\ and maps \( \psi_\alpha : V_\alpha \rightarrow U_\alpha \) where \( V_\alpha \) is an open subset of \( \mathbb{R}^n \) such that the following properties are satisfied

1. \( \cup_\alpha U_\alpha = M \)
2. \( \psi_\alpha : V_\alpha \rightarrow U_\alpha \) is a bijection for every \( \alpha \)
3. For any \( \alpha, \beta \) the set \( U_{\alpha \beta} = \psi_\alpha^{-1}(U_\alpha \cap U_\beta) \) is open in \( \mathbb{R}^n \)
4. For any \( \alpha, \beta \) the map \( \psi_\beta^{-1} \circ \psi_\alpha : U_{\alpha \beta} \rightarrow U_\beta \) is smooth.

The maps \( \psi_\alpha : V_\alpha \rightarrow U_\alpha \) are called local parameterizations.

The inverse maps \( \varphi_\alpha = \psi_\alpha^{-1} : U_\alpha \rightarrow V_\alpha \) are called local charts or local coordinates.

Two atlases on \( M \) are said to define the same smooth structure if their union is still a smooth atlas on \( M \).

**Examples**

- \( M = \mathbb{R}^n, \mathcal{A} = \{1\}, U_1 = V_1 = \mathbb{R}^n, \psi_1 = id \).
- \( M = \mathbb{R}, \mathcal{A} = \{1\}, U_1 = V_1 = \mathbb{R}, \psi_1 : \mathbb{R} \rightarrow \mathbb{R} \) is given by \( \varphi(x) = x^3 \).
- \( M = \Gamma_f : \mathbb{R}^n \rightarrow \mathbb{R}^k \). ( here \( \Gamma_f = \{(x, y) \in \mathbb{R}^{n+k} \mid y = f(x)\} \). \( \mathcal{A} = \{1\}, U_1 = \mathbb{R}^n, V_1 = M, \psi(x) = (x, f(x)) \)
- \( S^n = \{(x_0, x_1, \ldots, x_n) \mid \sum x_i^2 = 1\} \). For each \( i \) set \( U_i^+ = \{(x_0, x_1, \ldots, x_n) \in S^n \mid x_i > 0\} \) and \( U_i^- = \{(x_0, x_1, \ldots, x_n) \in S^n \mid x_i < 0\} \).

Let \( V_i^+ = \{(u_0, \ldots, u_{n-1}) \mid \sum u_i^2 < 1\} \) and let \( \psi_i^+ : V_i^+ \rightarrow U_i^+ \) be given by \( \psi_i(u) = (u_0, \ldots, u_{i-1}, \pm \sqrt{1 - |u|^2}, u_i, \ldots, u_{n-1}) \). The inverse map is just the projection \( \varphi_i^+(u_0, \ldots, u_n) = (u_0, \ldots, \hat{u}_i, \ldots, u_n) \)

This collection is a smooth atlas on \( S^n \) (verify!).

If \( j < i \) then
\[
\psi_j^{-1} \circ \psi_j(u_0, \ldots, u_{n-1}) = (u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{i-1}, \pm \sqrt{1 - |u|^2}, u_i, \ldots, u_{n-1})
\]

- \( \mathbb{RP}^n \) is the set of lines through 0 in \( \mathbb{R}^{n+1} \). It can be equivalently described as the set of equivalence classes of points in \( \mathbb{R}^{n+1} \setminus \{0\} \) modulo the equivalence relation \((x_0, \ldots, x_n) \sim (x'_0, \ldots, x'_n)\) iff \((x'_0, \ldots, x'_n) = t(x_0, \ldots, x_n)\) for some \( t \neq 0 \). The equivalence class of \((x_0, \ldots, x_n)\) will be denoted by \([x_0 : x_1 : \ldots : x_n]\).

Let \( U_i \subset \mathbb{RP}^n \) be the set of lines of the form \( \mathbb{R}(x_0, \ldots, x_n) \) with \( x_i \neq 0 \) for any nonzero point. any such line intersects the hyperplane
\{x_i = 1\} in a single point. This gives us a bijective map \(\varphi_i: U_i \to \mathbb{R}^n\) given by \(\varphi_i([x_0 : x_1 : \ldots : x_n]) = (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})\) with the inverse given by
\[
\psi_i(u_0, \ldots, u_{n-1}) = [u_0 : \ldots : u_{i-1} : 1 : u_i : \ldots : u_n].
\]
For \(j < i\) we have \(\varphi_j \circ \psi_i(u_0, \ldots, u_n) = \varphi_j([u_0 : \ldots, u_{i-1} : 1 : u_i : \ldots : u_{n-1}]) = (\frac{u_0}{u_j}, \ldots, \frac{u_{i-1}}{u_j}, \frac{u_{i+1}}{u_j}, \ldots, \frac{u_n}{u_j}).\)

This map is smooth on the open set \(\{u_j \neq 0\} = \psi_i^{-1}(U_j)\). A similar formula holds in the case \(j > i\). This gives a smooth atlas on \(\mathbb{R}^n\).

**Example 0.0.3.** Different smooth structures on \(\mathbb{R}\). The first structure is given by \(U_1 = V_1 = \mathbb{R}\) with \(\psi: \mathbb{R} \to \mathbb{R}\) given by the identity map \(\psi_1(x) = x\).

The second smooth structure is given by \(\tilde{U}_1 = \tilde{V}_1 = \mathbb{R}\) with \(\tilde{\psi}: \mathbb{R} \to \mathbb{R}\) given by \(\tilde{\psi}_1(x) = x^3\).

These smooth structure are distinct because these two atlases together do not form a smooth atlas since \(\tilde{\psi}_1^{-1} \circ \psi_1(x) = \sqrt[3]{x}\) is not smooth at zero.

**Definition 0.0.4.** Let \(X\) be a generalized smooth \(n\)-dimensional manifold with an atlas \(\{\psi_i: \mathbb{R}^n \to U_i\}_{\alpha \in \mathcal{A}}\). A subset \(U\) of \(X\) is called open if \(\psi^{-1}_\alpha(U)\) is an open subset of \(\mathbb{R}^n\) for any \(\alpha\).

It's easy to see that open sets satisfy the following properties:
- \(X\) and \(\emptyset\) are open.
- \(U_\alpha\) is open for any \(\alpha\).
- Union of any collection of open sets is open.
- Intersection of finitely many open sets is open.

**Definition 0.0.5.** A generalized smooth manifold \(M^n\) is called Hausdorff for any two distinct points \(p_1, p_2 \in X\) there exist open sets \(W_1, W_2 \subset X\) such that \(p_1 \in W_1, p_2 \in W_2\) and \(W_1 \cap W_2 = \emptyset\).

**Definition 0.0.6.** A generalized smooth manifold \(M^n\) is called a smooth manifold if it is Hausdorff and it admits a countable atlas \(\{\psi_i: \mathbb{R}^n \to U_i\}_{\alpha \in \mathcal{A}}\).

**Example 0.0.7.** Let \(I_1 = (-1, 1) \times \{1\}, I_2 = (-1, 1) \times \{2\}\). Let \(X\) be the space obtained from \(I_1 \cup I_2\) by identifying points \((x, 1)\) with \((x, 2)\) for all \(x \neq 0\). Let \(\pi: I_1 \cup I_2 \to X\) be the natural projection map and let \(\psi_i: (-1, 1) \to X\) be given by \(\psi_i(x) = \pi(x, i)\). This gives a smooth atlas on \(X\) with the transition map \(\psi_2^{-1} \circ \psi_1\) equal to the identity map of \((-1, 1) \setminus \{0\}\) to itself. Thus \(X\) is a generalized 1-dimensional manifold. However it is not Hausdorff (why?) and hence is not a smooth manifold.

**Example 0.0.8.** \(\mathbb{R}^n, S^n, \mathbb{R}P^n\) are Hausdorff and admit countable atlases. Hence they are smooth manifolds.