3. Exterior derivative (continued)

**Proposition 3.1.** Exterior differentiation on a manifold $M$ satisfies the following properties

a) Let $f : M \to \mathbb{R}$ be smooth. Then $df$ when viewed as exterior derivative of $f$ as a 0-form coincides with $df$ - differential of $f$.

b) $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is linear.

c) $d \circ d = 0$

d) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|\cdot|\eta|}\omega \wedge d\eta$

e) If $F : M^n \to N^m$ is smooth and $\omega$ is a form on $N$ then $F^*(d\omega) = d(F^*(\omega))$

Moreover, an operation $d : \Omega^*(M) \to \Omega^{*+1}(M)$ satisfying a)-d) is unique and must coincide with the exterior derivative.

**Proof.** The proof of the properties a)-d) is an immediate consequence of the definition and the fact that these properties hold for exterior derivatives on open subsets of $\mathbb{R}^n$.

To prove uniqueness, suppose $d$ is another operation satisfying a)-d).

Observe that locally any $\omega$ can be written as $\omega = \sum I \omega_I(x) dx^I$ where $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ and $x : U \to V \subset \mathbb{R}^n$ is a local coordinate chart on an open subset $U \subset M$. Then we have that $dx^I = dx^i$ by a) and hence $d(dx^I) = d(dx^i) = 0$ by c). Therefore, by repeatedly applying d) we get that $d(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = 0$. Therefore, by d) again we get that $d(\omega_I(x)dx^I) = d(\omega_I(x)) \wedge dx^I + \omega_I(x) \wedge d(dx^I) = dw_I(x) \wedge dx^I + 0 = d(\omega_I(x)dx^I)$. The general case follows by linearity of $d$ and $d$. \qed

4. De Rham cohomology

**Definition 4.1.** A form $\omega \in \Omega^*(M)$ is called closed if $d\omega = 0$.

A form $\omega \in \Omega^*(M)$ is called exact if $\omega = d\eta$ for some $\eta \in \Omega^{*-1}(M)$.

Since $d \circ d = 0$ it’s obvious that every exact form is closed. It’s natural to ask to what extent the converse holds. Let $B^k(M)$ be the set of all exact $k$-forms and let $Z^k(M)$ be the set of all closed $k$-forms. It’s obvious that $B^k(M)$, $Z^k(M)$ are vector spaces and by above $B^k(M) \subset Z^k(M)$.

**Definition 4.2.** Let $M^n$ be a smooth manifold, possibly with boundary. The $k$-th de Rham cohomology group of $M$ is defined to be the quotient group

$$H^k_{DR}(M) := Z^k(M)/B^k(M)$$

Since $B^k(M)$ is a vector subspace of $Z^k(M)$ the quotient $H^k_{DR}(M)$ is a vector space and not just a group.

By the definition that $H^k_{DR}(M) = 0$ iff every closed $k$-form is exact.
Example 4.4. Let $M = V$ be an open subset of $\mathbb{R}^2$. Then a 1-form $\omega$ on $V$ has the form $P(x,y)dx + Q(x,y)dy$. By definition, $\omega$ is exact iff $\omega = df$ for some smooth $f : V \to \mathbb{R}$, i.e., if $P(x,y)dx + Q(x,y)dy = \frac{\partial f}{\partial x}(x,y)dx + \frac{\partial f}{\partial y}(x,y)dy$ or $P(x,y) = \frac{\partial f}{\partial x}(x,y)$ and $Q(x,y) = \frac{\partial f}{\partial y}(x,y)$.

On the other hand $\omega$ is closed iff $0 = d\omega = d(P(x,y)dx + Q(x,y)dy) = (-\frac{\partial P}{\partial y}(x,y) + \frac{\partial Q}{\partial x}(x,y))dx \wedge dy$ or $-\frac{\partial P}{\partial y}(x,y) + \frac{\partial Q}{\partial x}(x,y) = 0$.

Thus, every closed 1-form on $V$ is exact iff for any smooth $P, Q : V \to \mathbb{R}$ satisfying $\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y)$ there exists a smooth $f : V \to \mathbb{R}$ such that $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$.

Exercise 4.4. Prove that $H^1_{DR}(\mathbb{R}^2) = 0$

Let $f : M \to N$ be a smooth map between manifolds. Since $F^* \circ d = d \circ F^*$ sends closed forms to closed forms and exact forms to exact forms. Therefore it induces a homomorphism $F^* : H^k_{DR}(N) \to H^k_{DR}(M)$ for any $k$.

Since $(G \circ F)^* = F^* \circ G^*$ and $Id_*^M = Id$ it follow that if $F : M \to N$ is a diffeomorphism then $F^* : H^k_{DR}(N) \to H^k_{DR}(M)$ is an isomorphism. We will see later that for $V = \mathbb{R}^2 \setminus \{0\}$ the form $\omega = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$ is closed but not exact. This will imply that $H^1_{DR}(\mathbb{R}^2 \setminus \{0\}) \neq 0$. Since $H^1(\mathbb{R}^2) = 0$ by the exercise above, this will show that $\mathbb{R}^2$ is not diffeomorphic to $\mathbb{R}^2 \setminus \{0\}$.

5. Orientation

5.1. Orientation on a vector space.

Definition 5.1. Let $V$ be a finite dimensional vector space. Let $e = (e_1, \ldots, e_n)$ and $e' = (e'_1, \ldots, e'_n)$ be two bases of $V$. We say that $e \sim e'$ if the transition matrix $A$ from $e$ to $e'$ has det $A > 0$. It’s easy to see that $\sim$ satisfies the following properties

- if $e \sim e'$ then $e' \sim e$;
- if $e \sim e'$ and $e' \sim e''$ then $e \sim e''$.

This means that $\sim$ is an equivalence relation on the set of all bases of $V$. We will call equivalence classes mod $\sim$ orientations on $V$. We will say that two bases $e, e'$ have the same orientation if they belong to the same equivalence class, i.e. the transition matrix from $e$ to $e'$ has positive determinant.

Lemma 5.2. Let $V$ be a finite dimensional vector space. Then there are precisely two possible ordinations on $V$.

Proof. Let $e = (e_1, \ldots, e_n)$ be a basis of $V$ and let $e' = (-e_1, e_2, \ldots, e_n)$. Since the transition matrix $A$ from $e$ to $e'$ has determinant $-1$ they define two different orientations on $V$. We claim that any other basis of $V$ is equivalent to either $e$ or $e'$: Let $e''$ be a basis of $V$. Let $B$ be the transition matrix from $e'$ to $e''$. Then the transition matrix from $e$ to $e''$ is $BA$ and $\det(BA) = \det(B) \cdot \det A = -\det B$. This means that $\det B$ and $\det(BA)$ have opposite signs, and thus one of them is positive and the other is negative. Therefore $e'' \sim e$ or $e'' \sim e'$.
We’ll call the two distinct orientations on \( V \) opposite or negative to each other. If \( \epsilon \) is an orientation and \( e = (e_1, \ldots, e_n) \) is a basis we put \( \epsilon(e) = +1 \) if \( e \) is positively oriented with respect to \( \epsilon \) and we put \( \epsilon(e) = -1 \) if \( e \) is negatively oriented with respect to \( \epsilon \).

\( \mathbb{R}^n \) has a canonical orientation defined by the canonical basis \((e_1, \ldots, e_n)\) of \( \mathbb{R}^n \).

Orientations on \( V \) correspond to orientations on \( \mathcal{A}^n(V) \cong \mathbb{R} \) as follows.

Let \( w \in \mathcal{A}^n(V) \) be a nonzero alternating \( n \)-tensor. It defines an orientation \( \epsilon_w \) as follows:

Given a basis \( e = (e_1, \ldots, e_n) \) we’ll say that \( e \) is positively oriented iff \( w(e_1, \ldots, e_n) > 0 \). It’s easy to see that this defines an orientation on \( V \). It’s also obvious that if \( w' = \lambda w \) with \( \lambda \neq 0 \) then \( w \) and \( w' \) define the same orientation iff \( \lambda > 0 \).

5.2. Orientation on manifolds. Let \( M^n \) be a smooth \( n \)-dimensional manifold (possibly with boundary).

**Definition 5.3.** An orientation \( \epsilon \) on \( M^n \) is a choice of orientation \( \epsilon(p) \) on \( T_pM \) for all \( p \in M \).

An orientation \( \epsilon \) is called continuous if for any \( p \in M \) there exists an open set \( U \subset M \) containing \( p \) and a collection of continuous vector fields \( X_1, \ldots, X_n \) on \( U \) such that \( X_1(q), \ldots, X_n(q) \) is a basis of \( T_qM \) for any \( q \in U \) and \( \epsilon(X_1(q), \ldots, X_n(q)) = +1 \) for any \( q \in U \).

A manifold \( M \) is called orientable if it admits a continuous orientation.

**Exercise 5.4.** Prove that an orientation \( \epsilon \) is continuous if and only if it’s smooth, i.e. for any \( p \in M \) there exists an open set \( U \subset M \) containing \( p \) and a collection of smooth vector fields \( X_1, \ldots, X_n \) on \( U \) such that \( X_1(q), \ldots, X_n(q) \) is a basis of \( T_qM \) for any \( q \in U \) and \( \epsilon(X_1(q), \ldots, X_n(q)) = +1 \) for any \( q \in U \).

From now on we will only consider continuous orientations. The relation between orientations and nonzero alternating \( n \)-vectors on a fixed vector space naturally carries over to manifolds as follows.

Suppose \( \omega \) is a smooth \( n \)-form on \( M^n \) such that \( \omega(p) \neq 0 \) for any \( p \in M \). Then \( \omega \) defines an orientation \( \epsilon_\omega \) on \( M \) as follows. Given \( p \in M \) and a basis \( v_1, \ldots, v_n \) of \( T_pM \) we say that it’s positively oriented iff \( \omega(p)(v_1, \ldots, v_n) > 0 \).

**Lemma 5.5.** \( \epsilon_\omega \) is continuous.

**Proof.** Let \( p \in M \) be any point. Let \( U \) be a coordinate ball containing \( p \), so \( U \) is diffeomorphic to an open ball \( B(0,1) \) in \( \mathbb{R}^n \) under some local coordinate map \( x: U \rightarrow \mathbb{R}^n \). Let \( X_i(q) = \frac{\partial}{\partial x_i}(q) \). Then \( f(q) = \omega(X_1(q), \ldots, X_n(q)) \) is smooth on \( U \). Since \( f(q) \neq 0 \) for any \( q \), by the Intermediate Value Theorem we must have that \( f(q) > 0 \) for all \( q \in U \) or \( f(q) < 0 \) for all \( q \in U \). In the first case this gives the required collection of continuous vector fields on \( U \). In the second case the same works after changing \( X_1 \) to \(-X_1\). \( \square \)
Next we will show that the converse also holds, i.e. every continuous orientation is equal to $\epsilon_\omega$ for some nowhere zero $\omega \in \Omega^n(M^n)$.

**Lemma 5.6.** Let $\epsilon$ be a continuous orientation on a smooth manifold $M^n$. Then there exists a smooth form $\omega \in \Omega^n(M)$ such that $\omega(p) \neq 0$ for any $p \in M$ and $\epsilon = \epsilon_\omega$.

**Proof.** Let $\epsilon$ be a continuous orientation on $M$.

We will use the following terminology. Let $\omega$ be a smooth $n$-form on an open subset $U \subset M$. We will say that $\omega$ is positive on $U$ if $\omega(p)(v_1, \ldots, v_n) > 0$ for any $p \in U$ and any positive basis $v_1, \ldots, v_n$ of $T_p M$. We need to prove that there exists a positive form on $U = M$.

Observe that if $\omega_1, \ldots, \omega_m$ are positive forms on $U$ and $\phi_1, \ldots, \phi_m : U \to \mathbb{R}$ are smooth functions such that $\phi_i \geq 0$ on $U$ and $\sum_i \phi_i > 0$ on $U$ then $\sum_i \phi_i \omega_i$ is positive on $U$.

For any $p \in M$ let $U_p$ be an open set containing $p$ such that there exist $n$ smooth vector fields $X_1, \ldots, X_n$ on $U_p$ such that $X_1(q), \ldots, X_n(q)$ is a positive basis of $T_q M$ for any $q \in U_p$. Let $X_1(q), \ldots, X_n(q)$ be the dual basis of $T_q^* M$. Then $X_1, \ldots, X_n$ are smooth forms on $U$ (why?). Let $\omega_p = X_1 \wedge \ldots \wedge X_n$. Then it’s a smooth positive form on $U$. Take a partition of unity $\{\phi_i\}$ subordinate to the cover $\{U_p\}_{p \in M}$ of $M$. Then supp $\phi_i \subset U_{p_i}$ for some $p_i$ and by the observation above $\omega = \sum_i \phi_i \omega_i$ is positive on all of $M$. \qed