(1) (8 pts) Give the following definitions
(a) A tangent vector to a smooth manifold
(b) A smooth manifold with boundary.

Solution
(a) A tangent vector to a smooth manifold $M$ at a point $p$ is a map $v: C^\infty(M) \to \mathbb{R}$ satisfying the following conditions
(i) $v$ is linear, i.e. $v(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 v(f_1) + \lambda_2 v(f_2)$ for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and $f_1, f_2 \in C^\infty(M)$.
(ii) $v(f \cdot g) = v(f)g(p) + f(p)v(g)$ for any $f, g \in C^\infty(M)$.
(b) A smooth manifold with boundary is a generalized smooth manifold with boundary which is Hausdorff and admits a countable atlas. A generalized smooth manifold with boundary is a set $M$ with a collection of maps $\{\psi_\alpha: V_\alpha \to U_\alpha\}_{\alpha \in \mathcal{A}}$ where $V_\alpha \subset \mathbb{H}^n$, $U_\alpha \subset M$, such that the following conditions are satisfied
(i) $\cup_\alpha U_\alpha = M$;
(ii) $\psi_\alpha: V_\alpha \to U_\alpha$ is 1-1 and onto;
(iii) $V_{\alpha\beta} = \psi_\alpha^{-1}(U_\beta)$ is open in $\mathbb{H}^n$ for any $\alpha, \beta$.
(iv) $\psi_\beta^{-1} \circ \psi_\alpha: V_{\alpha\beta} \to V_{\beta\alpha}$ is smooth for any $\alpha, \beta \in \mathcal{A}$.

(2) (10 pts) Let $f: \mathbb{RP}^n \to \mathbb{R}$ be given by $f([x_0 : x_1 : \ldots : x_n]) = \frac{x_0^2}{x_0^2 + x_1^2 + \ldots + x_n^2}$.
(a) Show that $f$ is well defined and smooth.
(b) Show that the set $\{f = 1/2\}$ is nonempty and carries a natural structure of a manifold of dimension $n - 1$.

Solution
(a) Let $\tilde{f}: \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}$ be given by $\tilde{f}(x_0, x_1, \ldots, x_n) = \frac{x_0^2}{x_0^2 + x_1^2 + \ldots + x_n^2}$. Then for any $\lambda \neq 0$ and any $x \in \mathbb{R}^{n+1}\setminus\{0\}$ we have $\tilde{f}(\lambda \cdot x) = \frac{\lambda^2 x_0^2}{\lambda^2 x_0^2 + \lambda^2 x_1^2 + \ldots + \lambda^2 x_n^2} = \frac{x_0^2}{x_0^2 + x_1^2 + \ldots + x_n^2} = \tilde{f}(x)$ which means that $f$ is well defined. To see that $f$ is smooth consider its representation in standard coordinate charts $\phi_i: \mathbb{R}^n \to \mathbb{RP}^n$ ($i = 0, \ldots, n$) given by $\phi_i(x_1, \ldots, x_n) = [x_1 : \ldots : x_i : 1 : x_{i+1} : \ldots : x_n]$. For $i < n$ we have $f \circ \phi_i(x_1, \ldots, x_n) = \frac{x_0^2}{x_1^2 + \ldots + x_n^2}$ is smooth in $x$. And for $i = n$ we have $f \circ \phi_n(x_1, \ldots, x_n) = \frac{1}{x_1^2 + \ldots + x_n^2 + 1}$ is also smooth in $x$.
Thus $f$ is smooth.
(b) $f(1 : 0 : \ldots : 0 : 1) = 1/2$ and hence $\{f = 1/2\}$ is nonempty. To see that $\{f = 1/2\}$ carries a natural structure of a manifold of dimension $n - 1$ it’s sufficient to check that $1/2$ is a regular value of $f$. Since the target is 1-dimensional it’s enough to show that $df_p \neq 0$ for any $p \in \{f = 1/2\}$.
We’ll check it for points lying in $U_i = \phi_i(\mathbb{R}^n)$ for all $i = 0, \ldots, n$. 
For $i < n$ we have $g_i(x) = f \circ \phi_i(x_1, \ldots, x_n) = \frac{x_i^2}{x_1^2 + \ldots + x_n^2}$.

We compute $\frac{\partial g_i}{\partial x_n}(x) = \frac{2x_i(1+x_1^2+\ldots+x_{n-1}^2)}{(x_1^2+\ldots+x_n^2)^2} \neq 0$ unless $x_n = 0$. However, if $x_n = 0$ then $g_i(x) = 0 \neq 1/2$ and hence $dg_i(x) \neq 0$ for any $x$ satisfying $g_i(x) = 1/2$.

Similarly, for $i = n$ we have $g_n(x) = \frac{1}{x_1^2 + \ldots + x_n^2 + 1}$. We compute

$$\frac{\partial g_i}{\partial x_i}(x) = \frac{-2x_i}{x_1^2 + \ldots + x_n^2 + 1}$$

Thus $dg_n(x) = 0$ only if all $x_i = 0$, i.e. $x = 0$. However, $g_n(0) = \frac{1}{1} = 1 \neq 1/2$ and hence we also have $dg_n(x) \neq 0$ for any $x$ satisfying $g_n(x) = 1/2$.

Altogether this means that $df_p \neq 0$ for any $p$ satisfying $f(p) = 1/2$ i.e. $1/2$ is a regular value of $f$.

(3) (12 pts) Mark True or False. You DO NOT need to justify your answers.

Let $M, N$ be smooth manifolds.

(a) A submersion $f: M \to N$ is onto.
Answer: False. E.g. take the inclusion $i: (-1,1) \to \mathbb{R}$.

(b) A smooth embedding $f: M \to N$ is a closed map.
Answer: False. Same example as in (a).

(c) If a smooth map $f: M \to N$ is injective then it’s an immersion.
Answer: False. E.g. $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.

(d) A local diffeomorphism which is 1-1 and onto is a diffeomorphism.
Answer: True.

(e) $S^1$ is diffeomorphic to $\mathbb{RP}^1$.
Answer: True.

(f) Composition of two maps of constant rank has constant rank.
Answer: False. E.g. take $f: (-1,1) \to \mathbb{R}^2$ given by $f(x) = (x, \sqrt{1-x^2})$ and $g: \mathbb{R}^2 \to \mathbb{R}$ given by $g(x,y) = y$.
Both maps have constant rank 1. However, $h(x) = g \circ h(x) = \sqrt{1-x^2}$ does not have constant rank.

(4) (10 pts) Let $M^n, N^m$ be smooth manifolds such that $n > m$. Let $f: M^n \to N^m$ be a smooth map.

Prove that $f$ is not 1-1.

**Hint:** Use the constant rank theorem.

**Solution**

Let $p \in M$ be a point where $r = \text{rank} \ df_p$ is maximal possible. Obviously, $r \leq m < n$. By reducing domain and range to co-ordinate charts we can assume that $M = U$ is an open subset in $\mathbb{R}^n$ and $N = \mathbb{R}^m$. By permuting co-ordinates we can assume that the matrix $[\frac{\partial f_i}{\partial x_j}(p)]_{i,j=1,\ldots,r}$ is invertible.
By continuity of determinants the same holds true for points \( x \) near \( p \). Thus, \( df_x \) has rank \( \geq r \) for \( x \) near \( p \). But since \( r \) was chosen to be maximal possible we actually have rank \( df_x = r \) for \( x \) near \( p \).

Thus, by the constant rank theorem after a change of coordinates near \( p \) on the domain and the target we can assume that \( f \) has the form \( f(y_1, \ldots, y_n) = (y_1, \ldots, y_r, 0, \ldots, 0) \) which is not 1-1 since \( r < n \). □

(5) (10 pts)
(a) Let \( f : M \to \mathbb{R} \) be smooth and let \( V \) be a smooth vector field on \( M \). Suppose \( V(p)(f) > 0 \) for some \( p \) in \( M \).

Prove that there is an open set \( U \) containing \( p \) such that \( V(x)(f) > 0 \) for any \( x \in U \).

(b) Let \( f : M \to \mathbb{R} \) be smooth. Suppose for every \( p \in M \) there exists \( v \in T_p M \) such that \( v(f) > 0 \).

Prove that there exists a smooth vector field \( V \) on \( M \) such that \( V(p)(f) > 0 \) for every \( p \in M \).

Hint: Use partition of unity.

Solution

(a) In some local coordinates \( V \) has the form \( V(x) = \sum_{i=1}^{n} v_i(x) \frac{\partial}{\partial x_i} \) where \( v_i(x) \) are smooth functions.

Then \( g(x) = V(x)(f) = \sum_{i=1}^{n} v_i(x) \frac{\partial f}{\partial x_i}(x) \) is smooth in \( x \). In particular, it’s continuous and since \( g(p) > 0 \), by continuity, there exists an open set \( U \) containing \( p \) such that \( g(x) > 0 \) for any \( x \in U \). □

(b) Let \( p \in M \) be any. We are given that there exists \( v \in T_p M \) such that \( v(f) > 0 \).

In some local coordinates \( x \) it has the form \( v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i} \) for some \( v_1, \ldots v_n \in \mathbb{R} \). Extend \( v \) to a smooth vector field \( V_p \) on a small open set \( U_p \) containing \( p \) by the formula \( V_p(x) = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i} \) for \( x \in U_p \).

The collection \( \{W_p\}_{p \in M} \) is an open cover of \( M \). Let \( \{\phi_i\}_{i=1}^{\infty} \) be a partition of unity subordinate to this cover so that \( \text{supp} \phi_i \subset W_{p_i} \).

Let \( V = \sum_i \phi_i \cdot V_{p_i} \). We claim that \( V \) satisfies the required properties: \( V \) is obviously smooth and for any \( p \in M \) we have

\[
V(p)(f) = \sum_i \phi_i \cdot V_{p_i}(p)(f) \geq 0
\]

since all the terms in the sum are nonnegative. Moreover, for every \( p \) there is an \( i \) such that \( \phi_i(p) > 0 \). For that \( i \) we have that \( p \in \text{supp} \phi_i \subset W_{p_i} \) and therefore, \( V_{p_i}(p)(f) > 0 \). Thus, at least one term in the sum (1) is positive and hence, \( V(p)(f) > 0 \). □