1. Weak vs strong convergence
Consider the sequence of functions \( f_k(x) = \frac{1}{k} \sin(k\pi x) \), where \( x \in (0,1) \).

- Show that \( f_k \) is uniformly bounded in \( W^{1,2}((0,1)) \).
- Show that \( f_k \rightharpoonup 0 \) in \( W^{1,2}((0,1)) \).
- Show that \( f_k \) does not converge strongly in \( W^{1,2}((0,1)) \).

2. Issues with a naive approach to the Plateau problem
Let \( \Gamma \subset \mathbb{R}^3 \) be a simple closed curve. Consider the class of functions

\[
\mathcal{F}_\Gamma = \{ u \in W^{1,2}(D^2, \mathbb{R}^3) | \quad u|_{\partial D} \in C^0(\partial D, \mathbb{R}^3) \text{ is a weakly monotone parametrization of } \Gamma \},
\]
\hspace{1cm} (0.1)

and the energy functional \( E : \mathcal{F}_\Gamma \to \mathbb{R} \),
\[
E[u] = \frac{1}{2} \int_{D^2} |\nabla u|^2 dx
\]
\hspace{1cm} (0.2)

Let \( \mathcal{G} \) be the Mobius group of the disc,
\[
\mathcal{G} = \left\{ \varphi(z) = e^{i\alpha} \frac{a + z}{1 + \bar{a}z} | \alpha \in \mathbb{R}/2\pi\mathbb{Z}, |a| < 1 \right\}.
\]
\hspace{1cm} (0.3)

- Prove that \( E[u] = E[u \circ \varphi] \) for all \( u \in \mathcal{F}_\Gamma, \varphi \in \mathcal{G} \).
- Let \( u \in \mathcal{F}_\Gamma \). Prove that there exists a sequence \( \varphi_k \in \mathcal{G} \) such that \( u \circ \varphi_k \) converges weakly in \( W^{1,2}(D^2, \mathbb{R}^3) \) to a constant map.

3. The \( q \)-Laplacian
Let \( \Omega \subset \mathbb{R}^n \) be a smooth domain, and let \( f : \bar{\Omega} \to \mathbb{R} \) be a smooth function. For \( 1 < q < \infty \), the \( q \)-Laplacian is the quasilinear 2nd order elliptic operator defined by
\[
\Delta_q u = \text{div}(|\nabla u|^{q-2}\nabla u).
\]
\hspace{1cm} (0.4)
• Prove the existence of a weak solution of the Dirichlet problem

\[-\Delta_q u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.\]  \hspace{1cm} (0.5)

• Prove the uniqueness of weak solution of (0.5). You may assume \( q \geq 2 \).

• Give an example of a nonsmooth solution of (0.5) for \( q \) and \( \Omega \) and \( f \) (smooth) of your choice.

4. A forth order variational elliptic equation

Let \( \Omega \subset \mathbb{R}^n \) be a smooth domain, and let \( f : \Omega \to \mathbb{R} \) be a smooth function. Consider the energy functional

\[E[u] = \int_{\Omega} \left( \frac{1}{2} |\nabla^2 u|^2 + fu \right) dx,\]  \hspace{1cm} (0.6)

where \( \nabla^2 \) denotes the Hessian, subject to the boundary conditions

\[u = 0 \text{ on } \partial \Omega \quad \text{and} \quad \nabla_n u = 0 \text{ on } \partial \Omega.\]  \hspace{1cm} (0.7)

• Suppose \( u \) is a smooth minimizer of \( E \). Prove that \( u \) satisfies the Euler-Lagrange equation

\[-\Delta^2 u = f.\]  \hspace{1cm} (0.8)

• Let \( \mathcal{F} = \{ u \in W^{2,2}(\Omega) | u \text{ satisfies (0.7) in the trace sense} \} \) and consider the energy (0.6) as a functional \( E : \mathcal{F} \to \mathbb{R} \). Prove the existence of a minimizer.

• Prove that any \( u \in \mathcal{F} \) with \( E[u] = \inf_{v \in \mathcal{F}} E[v] \) is a weak solution of (0.8).

• Prove that any \( u \in \mathcal{F} \) with \( E[u] = \inf_{v \in \mathcal{F}} E[v] \) is smooth.