Moser Iteration and $\varepsilon$-regularity

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The goal of these notes is to explain step by step how to prove the following $\varepsilon$-regularity theorem via Moser iteration:

**Theorem**

There are constants $\varepsilon > 0$ and $C < \infty$ with the following significance. If $n = 4$ and $u \in H^1(B_1)$ is a nonnegative function that satisfies

$$-\Delta u \leq u^2$$

weakly, then

$$\|u\|_{L^2(B_1)} \leq \varepsilon \Rightarrow \|u\|_{L^\infty(B_1/2)} \leq C\|u\|_{L^2(B_1)}.$$  \(2\)

**First step of the Moser iteration.** Choose a cutoff-function $0 \leq \eta \leq 1$ that is $1$ on $B_{3/4}$, has support in $B_1$, and satisfies $|\nabla \eta| \leq 8$. Multiplying (1) by $\eta^2 u$ and integrating by parts we obtain

$$\int_{B_1} \eta^2 |\nabla u|^2 \leq 2 \int_{B_1} \eta |\nabla u| |\nabla \eta| u + \int_{B_1} \eta^2 u^3.$$  \(3\)

Dealing with the first term on the right hand side by Young’s inequality and absorption, this gives the estimate

$$\frac{1}{2} \int_{B_1} \eta^2 |\nabla u|^2 dV \leq 128 \int_{B_1} u^2 + \int_{B_1} \eta^2 u^3.$$  \(4\)

For the last term, using Hölder’s inequality, the assumption that the energy on $B_1$ is less than $\varepsilon$, and the Sobolev-inequality, we get

$$\int_{B_1} \eta^2 u^3 \leq \left( \int_{B_1} u^2 \right)^{1/2} \left( \int_{B_1} (\eta u)^4 \right)^{1/2} \leq \varepsilon C_S^2 \int_{B_1} |\nabla (\eta u)|^2$$

$$\leq 2\varepsilon C_S^2 \int_{B_1} \eta^2 |\nabla u|^2 + 128\varepsilon C_S^2 \int_{B_1} u^2,$$

where $C_S < \infty$ is the local Sobolev constant on $B_1$. The main idea is that if we choose $\varepsilon$ so small that $2\varepsilon C_S^2 \leq \frac{1}{4}$ then the $\int \eta^2 |\nabla u|^2$ term can be absorbed, giving

$$\frac{1}{4} \int_{B_1} \eta^2 |\nabla u|^2 \leq 144 \int_{B_1} u^2$$  \(6\)
and using the Sobolev inequality we arrive at the $L^4$-estimate
\[ \|u\|_{L^4(B_{3/4})} \leq 24C_S\|u\|_{L^2(B_1)}. \]  
(7)

**General step of the Moser iteration.** Fix $\varepsilon \leq \frac{1}{5}C_S^{-2}$, where $C_S$ is the local Sobolev constant on $B_1$. Consider the sequence of radii $r_k = \frac{1}{2} + \frac{1}{2^k}$ interpolating between $r_1 = 1$ and $r_\infty = \frac{1}{2}$. We want to prove by induction an estimate of the form
\[ \|u\|_{L^{2k+1}(B_{r_{k+1}})} \leq C_k\|u\|_{L^{2k}(B_{r_k})}. \]  
(8)

The case $k = 1$ has already been established above (with $C_1 = 24C_S$). For general $k \geq 2$ we multiply (1) by $\eta_k^2 u^{2\alpha_k-1}$, where $\alpha_k = 2^{k-1}$, and $0 \leq \eta_k \leq 1$ is a cutoff function that equals 1 on $B_{r_k}$, has support in $B_{r_k}$, and satisfies $|\nabla\eta_k| \leq 2/(r_k - r_{k+1})$. After integration by parts, we obtain
\[ \frac{2\alpha_k - 1}{\alpha_k^2} \int_{B_{r_k}} \eta_k^2 |\nabla u^{\alpha_k}|^2 \leq \int_{B_{r_k}} \eta_k |\nabla u| \|\nabla\eta_k\| u^{2\alpha_k-1} + \int_{B_{r_k}} \eta_k^2 u^{2\alpha_k+1}. \]  
(9)

Dealing with the first term on the right hand side by Young’s inequality and absorption, this gives the estimate
\[ \frac{2\alpha_k - 1}{2\alpha_k^2} \int_{B_{r_k}} \eta_k^2 |\nabla u^{\alpha_k}|^2 \leq \frac{32\alpha_k^2}{2\alpha_k - 1} \int_{B_{r_k}} u^{2\alpha_k} + \int_{B_{r_k}} \eta_k^2 u^{2\alpha_k+1}. \]  
(10)

For the last term, using Hölder’s inequality, the Peter-Paul inequality, the Sobolev-inequality, and the estimate (7), we compute
\[ \int_{B_{r_k}} \eta_k^2 u^{2\alpha_k+1} \leq \left( \int_{B_{r_k}} \eta_k^4 u^{4\alpha_k} \right)^{1/4} \left( \int_{B_{r_k}} \eta_k^4 u^4 \right)^{1/4} \left( \int_{B_{r_k}} u^{2\alpha_k} \right)^{1/2} \]  
(11)
\[ \leq \delta_k C_S \int_{B_{r_k}} |\nabla(\eta_k u^{\alpha_k})|^2 + \frac{1}{4\delta_k} \left( \int_{B_{r_k}} \eta_k^4 u^4 \right)^{1/2} \int_{B_{r_k}} u^{2\alpha_k} \]  
\[ \leq 2\delta_k C_S^2 \int_{B_{r_k}} \eta_k^2 |\nabla u^{\alpha_k}|^2 + \left( 128\delta_k C_S^2 \alpha_k^2 + \frac{1}{4\delta_k} 24\alpha_k^2 \varepsilon^2 \right) \int_{B_{r_k}} u^{2\alpha_k}. \]

Choosing $\delta_k = (2\alpha_k - 1)/(8\alpha_k^2 C_S^2)$ the first term can be absorbed, giving
\[ \frac{2\alpha_k - 1}{4\alpha_k^2} \int_{B_{r_k}} \eta_k^2 |\nabla u^{\alpha_k}|^2 \leq \left( \frac{32\alpha_k^2}{2\alpha_k - 1} + 16(2\alpha_k - 1) + \frac{18\alpha_k^2}{2\alpha_k - 1} \right) \int_{B_{r_k}} u^{2\alpha_k}, \]  
(12)
and using the Sobolev inequality we arrive at the estimate (8), with
\[ C_k \leq (D2^{2k})^{1/2^k}, \]  
(13)
where $D < \infty$ is a universal constant (in fact $D = 100$ works). The product of the constants $C_k$ is bounded and sending $k \to \infty$ gives the desired estimate
\[ \|u\|_{L^\infty(B_{1/2})} \leq C\|u\|_{L^2(B_1)}. \]  
(14)