0.1 Grade é

5.1. Divide the chessboard into two connected parts so that the first part is 4 squares more than the second one but the second part contains 4 black squares more than the first one.
Part is connected if it stays in one piece; connection must be by a segment.

SOLUTION.

5.2. From Monday to Friday a worker painted a fence in 8 hours’ shifts. On Monday he worked twice as slow as during midweek (Tuesday, Wednesday, Thursday). On Friday he worked twice as fast as during midweek and finished his job after 6 hours. In the result he painted 300 meters more on Friday than on Monday. How long was the fence?
SOLUTION. On Friday the worker painted 4 times as fast as on Monday but since he worked only 6 hours instead of 8 he painted only 3 times more than on Monday. Since the difference between the parts of the fence painted on Friday and Monday
is 300 meters, the worker painted 150 m on Monday and 450 m on Friday. Each midweek day he painted twice as much as Monday, thus 300 m per day. Therefore, in total the worker painted $150 + 300 \times 3 + 450 = 1500$ m.

5.3. Determine the number of 4-digit numbers each composed of distinct digits with the first digit divisible by 2 and the sum of the first and the last digits divisible by 3.

**Solution.** The first digit could be 2, 4, 6, 8. If the first digit is 2, then the last one could be 1, 4, 7; if the first digit is 4, then the last one could be 2, 5, 8; if the first digit is 6, then the last one could be 0, 3, 9 (6 is excluded as the digits must be distinct); if the first digit is 8, then the last one could be 1, 4, 7. Therefore we have $4 \times 3 = 12$ choices for the first and the last digits. As two digits are already chosen we have only 8 choices for the second digit and 7 choices for the third digit. Hence, the total number of choices is $12 \times 8 \times 7 = 672$.

5.4. Simpsons family celebrates only those birthdays when one’s age equals to the sum of the digits of his/her birth year. Adam’s celebration was in 2013 and Betty’s celebration was in 2014. Who is older and by how many years?

**Solution.** Observe that age of each of both children can not exceed 28 years. (Indeed, no year up to 2014 has the sum of digits greater than 28). Therefore we are left to check the years from 1985 to 2013 for Adam and the years from 1986 to 2014 for Betty. Checking, we get that Adam could born either in 2010 or in 1992 while Betty could born either in 2006 or in 1988. Therefore, the possible answers are:

(a) Betty is 4 years older (she was born in 1988 while Adam in 1992 or she was born in 2006 while Adam in 2010);

(b) Betty is 22 years older (she was born in 1988 while Adam in 2010);

(c) Adam is 14 years older (he was born in 1992 while Betty in 2006);

5.5. Karlsson bought in cafeteria several crepes (25 rubles per piece) and several jars with honey (340 rubles per jar). When he told Smidge the total amount he spent Smidge was able to determine the number of crepes and the number of jars with honey. Can it happen that Karlsson spent more than 2000 rubles?

**Solution.** To prove we need to construct an example. Observe that the cost of 5 jars of honey equals the cost of 68 crepes and no smaller number of jars can be replaced by the integer number of crepes. (Indeed, the numbers 5 and 68 are relatively prime: they share no common divisor, except 1). Hence if Karlsson bought 4 jars of honey and 67 crepes he paid
340 \times 4 + 25 \times 67 = 3035 > 2000 \text{ rubbles and Smidge is able to uniquely determine the items he bought.}

5.6. Brothers found a treasure of gold and silver. They divided it so that each share was 100 kg. The oldest brother got 30 kg (more than anyone else) of gold and one fifth of all the silver. How much gold was there in the treasure?

SOLUTION. The oldest brother got 30 kg of gold and therefore 70 kg of silver. Therefore the total amount of silver is 350 kg. Then the total amount of gold is 100n - 350, where n is the number of the brothers. Since the oldest brother got more gold than anyone else, we have \((100n - 350)/n < 30 \implies n < 5\). Since \(350 \leq 100n\) (the amount of silver is less than amount of the treasure), \(n = 4\). Hence, the amount of gold is \(4 \cdot 100 - 350 = 50\) kg.

Grade R6

6.1. Divide the chessboard into two connected parts so that the first part is 6 squares more than the second one but the second part contains 6 black squares more than the first one.

Part is connected if it stays in one piece; connection must be by a segment.

Solution.

6.2. Simpsons family celebrates only those birthdays when one’s age equals to the sum of the digits of his/her birth year. Adam’s celebration was in 2013 and Betty’s celebration was in 2014. Who is older and by how many years?

Solution. Observe that age of each of both children can not exceed 28 years. (Indeed, no year up to 2014 has the sum of digits greater than 28). Therefore we are left to check the years from 1985 to 2013 for Adam and
the years from 1986 to 2014 for Betty. Checking, we get that Adam could
born either in 2010 or in 1992 while Betty could born either in 2006 or in
1988. Therefore, the possible answers are:

(a) Betty is 4 years older (she was born in 1988 while Adam in 1992 or she
was born in 2006 while Adam in 2010);
(b) Betty is 22 years older (she was born in 1988 while Adam in 2010);
(c) Adam is 14 years older (he was born in 1992 while Betty in 2006);

6.3. Determine the number of 5-digit numbers each composed of distinct
digits with the first digit divisible by 2 and the sum of the first and the last
digits divisible by 3.

Solution.
The first digit could be 2, 4, 6, 8. If the first digit is 2, then the last digit can
be 1, 4, or 7. If the first digit is 4, then the last one can be 2, 5, or 8. If the
first digit is 6, then the last one can be 0, 3, or 9 (6 is excluded as the digits
must be distinct). If the first digit is 8, then the last one can be 1, 4, or 7.
Therefore we have $4 \times 3 = 12$ choices for the first and the last digits. As
two digits are already chosen we have only 8 choices for the second digit,
7 choices for the third digit and 6 choices for the fourth digit. Hence, the
total number of choices is $12 \times 8 \times 7 \times 6 = 4032$.

6.4. At the beginning of year an exchange rate of US dollar to euro was 0.8.
An expert predicted that during this year an exchange rate euro to rubble
would increase by 8% while a rate US dollar to rubble would drop by 10%.
If his prediction is correct what would be an exchange rate of US dollar to
euro by the end of the year?

Solution. Suppose that at the beginning of the year one dollar was $d_1$
rubles and one euro was $e_1$ rubles. According to prediction, to the end
of the year a dollar will be $d_2 = 0.9d_1$ (drop by 10%) and an euro will be
$e_2 = 1.08e_1$ (increase by 8%) rubles. Then $d_2 : e_2 = 0.9d_1 : 1.08e_1 = \frac{5}{6}d_1 : e_1 = 2/3$. Therefore, if prediction is true, an exchange rate of dollar to euro
will be $2/3$.

6.5. Brothers found a treasure of gold and silver. They divided it so that
each share was 100 kg. The oldest brother got 30 kg (more than anyone
else) of gold and one fifth of all the silver. How much gold was there in the
treasure?

Solution. The oldest brother got 30 kg of gold and therefore 70 kg of silver.
Therefore the total amount of silver is 350 kg. Then the total amount of gold
is $100n - 350$, where $n$ is the number of the brothers. Since the oldest brother
got more gold than anyone else, we have $(100n - 350)/n < 30 \implies n < 5$. Since $350 \leq 100n$ (the amount of silver is less than amount of the treasure), $n = 4$. Hence, the amount of gold is $4 \cdot 100 - 350 = 50$ kg.
Grade R7

7.1. Two girls, Ann and Betty, thought of a number (each of her own). Then each girl wrote all the divisors of her number, Ann wrote 10 numbers and Betty wrote 9 numbers. How many distinct numbers were written on the board if the greatest number written twice was 50?

Solution. Since the greatest common divisor of Ann’s and Betty’s numbers is 50, all six divisors of 50 (1, 2, 5, 10, 25, 50) are written twice and there is no more common numbers. Therefore, the number of distinct numbers written by both girls is 10 + 9 − 6 = 13.

7.2. A closed broken line is constructed along the lines of a grid, with its total length equal to 36 cell sides. What is the maximal area bounded by this line?

Solution. Let $a$, $b$ be projections of the figure to $x$- and $y$-axis. Then the perimeter of this figure is at least $2(a + b)$; therefore, $a + b \leq 18$. Since the figure is inside rectangle with area $ab$, the area of the figure is at most $ab$. Therefore $ab \leq a(18 − a) \leq 9^2$ where the equality is achieved on $9 \times 9$-square. The latter follows from the fact that among all rectangles with the fixed perimeter, a square has the greatest area.

7.3. Consider a circle and three equal chords passing through one point. Prove that each chord is a diameter.

Solution. Consider two chords $AB$ and $CD$. Since they are equal, the corresponding arcs $ACB$ and $CBD$ are equal and therefore short arcs $AC$ and $BD$ are also equal. Then $\angle ADC = \angle BAD$ and $\triangle AQD$ is isosceles. Therefore, $Q$ is equidistant from $A$ and $D$.

In the case of three equal chords passing through the same point, the point $Q$ is equidistant from three different points of the circumference. Therefore, $Q$ must be a centre of a circle.
7.4. Brothers found a treasure of gold and silver. They divided it so that each share was 100 kg. The oldest brother got 25 kg (more than anyone else) of gold and one eighth of all the silver. How much gold was there in the treasure?

Solution. The oldest brother got 25 kg of gold and therefore 75 kg of silver. Therefore the total amount of silver is 600 kg. Then the total amount of gold is \(100n - 600\), where \(n\) is the number of the brothers. Since the oldest brother got more gold than anyone else, we have \((100n - 600)/n < 25 \implies n < 8\). Since \(600 \leq 100n\) (the amount of silver is less than amount of the treasure), \(n > 6\). Therefore, \(n = 6\) and the amount of gold is \(7 \cdot 100 - 600 = 100\) kg.

7.5. The distance between two villages \(A\) and \(B\) is 45 km. Three friends have two bicycles, the speed of a cyclist is 15 km/h and the speed of a hiker is 5 km/h. What is the minimal time needed for them to go from \(A\) to \(B\)? Two people cannot ride the same bike simultaneously and they cannot leave the bike on the road unattended.

Solution. Let \(X, Y\) and \(Z\) stand for three friends. Suppose \(M\) is a midpoint of \(AB\). Let \(X\) and \(Y\) ride to \(M\) (1.5 h). Let \(X\) continue to destination by foot (4.5 h) while \(Y\) wait for \(Z\) who is hiking to point \(M\). After 4.5 hours \(Z\) reaches \(M\) and both \(Z\) and \(Y\) ride to \(B\). It takes them another 1.5 hours to reach their destination. Exactly at the same moment \(X\) also arrives to \(B\). Suppose the point \(M\) is not in the middle of \(AB\). It is clear that for either \(Z\) or \(X\) the walking distance will be greater then the half of the whole distance and the shorter distance for riding does no compensate the extra walking time.

Answer: the minimal time given this scenario is 6 hours.

7.6. Lev took two natural numbers and added their sum to their product, getting 1000. What numbers might that be? Find all possible pairs.

Solution. Let us numbers be \(x\) and \(y\). Then \(xy + x + y = 1000\); adding 1 to both sides we get \((x + 1)(y + 1) = xy + x + y + 1 = 1001\). Therefore \(x + 1 \geq 2\) and \(y + 1 \geq 2\) must be divisors of 1001 = 7 · 11 · 13. Then either \(x + 1 = 7, y + 1 = 11 \cdot 13 = 143\), or \(x + 1 = 11, y + 1 = 7 \cdot 13 = 91\), or \(x + 1 = 13, y + 1 = 7 \cdot 11 = 77\) or the other way around. Hence, all possible pairs are (6,142), (10, 90), (12,76) and (142,6), (90,10), (76,12).
Grade R8

8.1. Two children, Ann and Betty, thought of a number (each of her own). Then each girl wrote all the divisors of her number, Ann wrote 10 numbers and Betty wrote 9 numbers. How many distinct numbers were written if both students wrote number 6?

Solution. We know that the number $a$ written in prime factorization form $a = p_1^{n_1}p_2^{n_2}\cdots p_s^{n_s}$ has $(n_1 + 1)(n_2 + 1)\cdots(n_s + 1)$ distinct divisors. Since 6 is a divisor of both Ann’s and Betty’s numbers, there are at least two common prime divisors (2 and 3) on lists of both girls. On the other hand, in prime factorization of Ann’s number can not be more than two different primes. Indeed 10 can be represented as product of at least two numbers in the unique way, namely $10 = 2 \cdot 5$. Therefore Ann’s number is either $2^4 \cdot 3 = 48$ or $2 \cdot 3^4 = 162$. Betty’s number has 9 distinct divisors and 2 and 3 are among them. By similar arguments, since $9 = 3 \cdot 3$ Betty’s number is $2^2 \cdot 3^2 = 36$. It is easy to check that for either Ann’s number, the number of distinct numbers on lists of both girls is 13.

8.2. Consider a circle and three equal chords passing through one point. Prove that each chord is a diameter.

![Diagram of a circle with chords]

Solution. Each chord passing through point $Q$ is divided by $Q$ into short and long parts. By Power Point Theorem $AQ \times BQ = CQ \times DQ$. Therefore if $AB = CD$ then we have fixed sum and product of the segments of the chords and therefore, short parts of the chords are equal (long parts are also equal). Therefore $Q$ is equidistant from three different points on the circumference and must be a centre of the circle.

Remark. We also proved that short and long parts are the same.

Remark. Another solution see in 7.3.

8.3. Brothers found a treasure of gold and silver. They split it so that each share was 100kg. The oldest brother got $\frac{1}{5}$ of all gold, and $\frac{1}{7}$ of all silver.
The youngest brother got 1/7 of all gold. What part of all silver did the youngest brother get?

**Solution.** Let $x$ be amount of gold in the treasure. Then $100n - x$ is of amount of silver where $n$ is the number of brothers. Then the oldest brother got $\frac{1}{5}x + \frac{1}{7}(100n - x) = 100$ (kg) of the treasure and therefore $x = 1750 - 250n$. Then the total amount of silver is $350n - 1750$. Assuming that both amounts of gold and silver are positive we get $n = 6$. Therefore 250 and 350 are amounts of gold and silver in the treasure respectively. Hence the youngest brother got $250/7$ kg of gold and therefore $100 - 250/7 = 450/7$ of silver, which constitutes $1/7 \times 450/350 = 9/49$ of the total amount of silver.

**8.4.** The distance between two villages $A$ and $B$ is 45 km. Three friends have two bicycles, the speed of a cyclist is 15 km/h and the speed of a hiker is 5 km/h. What the minimal time is needed for them to go from $A$ to $B$? Two people cannot ride the same bike simultaneously and they cannot leave the bike on the road unattended.

**Solution.** Let $X$, $Y$ and $Z$ stand for three friends. Suppose $M$ is a midpoint of $AB$. Let $X$ and $Y$ ride to $M$ (1.5 h). After they reached $M$ let $X$ hike to destination (4.5 h) while $Y$ wait for $Z$ who is still hiking to $M$. After 4.5 hours on his way $Z$ reaches $M$ and both $Z$ and $Y$ ride to $B$. It takes them another 1.5 hours to reach their destination. Exactly at the same moment $X$ also arrives to $B$. Suppose the point $M$ is not in the middle of $AB$. It is clear that for either $Z$ or $X$ the walking distance will be greater then the half of the whole distance and the shorter distance for riding does no compensate the extra walking time.

**Answer:** the minimal time given this scenario is 6 hours.

**Remark.** (a) Assume that the cyclist can drag the second bike in the tow. We can check that the following scenario requires less time (5 hours).

$X$ and $Y$ ride to point $M$, 30 km away from $A$ (2 hours). Then $X$ hikes to $B$ (another 3 hours) while $Y$ with the second bike in a tow rides back to met $Z$. In one hour he meets $Y$ (who hiked 15 km to this moment) and they both ride 30 km to $B$ (2 hours). They spent 5 hours for the whole trip, the same as $X$ (who walked the same distance and rode the same distance as $Z$).

(b) Assume that one can leave a bike without supervision. It is pretty much clear that walking time for each friend must be the same. Indeed, if one walked more time than some other, then he would spend more time on whole
trip since the shorter riding time can not compensate the longer walking time. This implies that each of three friends must walk $1/3$ of the distance and ride $2/3$ of the distance. It can be achieved as follows.

$Z$ hikes to a point $M$ that is 15 km away from $A$, while $X$ rides to $M$ and leaves his bike to be picked up by $Z$, who will ride directly to $B$. Meanwhile, $Y$ continues to ride to a point $N$ that is 15 km away from $B$ and then hikes to $B$, leaving his bike to be picked up by $X$ who will ride to $B$. In this scenario the whole trip will take 5 hours.

8.5. Karlsson bought in cafeteria several crepes (25 rubles per piece) and several jars with honey (340 rubles per jar). When he told Smidge the total amount he spent Smidge was able to determine the number of crepes and the number of jars with honey. Can it happen that Karlsson spent more than 2000 rubles?

**Solution.** To prove we need to construct an example. Observe that the cost of 5 jars of honey equals the cost of 68 crepes and no smaller number of jars can be replaced by the integer number of crepes. (Indeed, the numbers 5 and 68 are relatively prime: they share no common divisor, except 1). Hence if Karlsson bought 4 jars of honey and 67 crepes he paid $340 \times 4 + 25 \times 67 = 3035 > 2000$ rubbles and Smidge is able to uniquely determine the items he bought.

8.6. A closed broken line is constructed along the lines of a grid, with its total length equal to 2014 cell sides. What is the maximal area bounded by this line?

**Solution.** Let $a$, $b$ be projections of the figure to $x$- and $y$-axis. Then the perimeter of this figure is at least $2(a + b)$; therefore, $a + b \leq 1007$. Since the figure is inside rectangle with area $ab$, the area of the figure is at most $ab$. Among all rectangles with the fixed perimeter, a square has the greatest area (we use it as a well known fact); however for rectangles on a grid it is not possible to get a square if half-perimeter is odd. Therefore, the maximal area will have a rectangle which is almost a square: one side is $\lceil 1007/2 \rceil = 503$ and another should be $\lfloor 1007/2 \rfloor = 504$. Then the the maximal area is $503 \times 504 = 253512$.

**Remark.** Proof that the maximal area will be achieved on “almost a square” can be checked by brute force.
Grade R9

9.1. Given a convex pentagon. For each pair of its diagonals intersecting inside consider the smallest angle between them. Find all possible values of the sum of all these five angles.

Solution.

Let $\Sigma$ be the required sum. Consider convex pentagon $FGHIJ$ formed by diagonals. It is obvious that $\Sigma$ cannot be larger than the sum of adjacent angles of $FGHIJ$ which is $5 \cdot 180^\circ - S = 360^\circ$ where $S = 3 \cdot 180^\circ$ is the sum of inner angles. Therefore $\Sigma \leq 360^\circ$. If all inner angles of $FGHIJ$ are obtuse, $\Sigma = 360^\circ$.

Let us transform case (a) by moving points $E$ and $D$ far away to the left and $B$ and $C$ to the right (case (b)). We can see that $\angle JFG = \alpha$ and $\angle IHG = \beta$ can be made arbitrarily small. In comparison to case (a) the value of $\Sigma$ is decreased by the sum of $(180^\circ - 2\alpha)$ and $(180^\circ - 2\beta)$ so in (b) $\Sigma = 2(\alpha + \beta)$. Therefore it can be made arbitrarily small. Hence, the answer is $\Sigma \in (0^\circ, 360^\circ]$.

9.2. Two children, Ann and Betty, thought of a number (each of her own). Then each girl wrote all the divisors of her number, Ann wrote 10 numbers and Betty wrote 9 numbers. How many distinct numbers were written if both students wrote number 6?

Solution. We know that the number $a$ written in prime factorization form $a = p_1^{n_1}p_2^{n_2} \cdots p_s^{n_s}$ has $(n_1 + 1)(n_2 + 1) \cdots (n_s + 1)$ distinct divisors. Since 6 is a divisor of both Ann’s and Betty’s numbers, there are at least two common prime divisors (2 and 3) on lists of both girls. On the other hand, in prime factorization of Ann’s number can not be more than two different primes. Indeed 10 can be represented as product of at least two numbers in the unique way, namely $10 = 2 \cdot 5$. Therefore Ann’s number is either
$2^4 \cdot 3 = 48$ or $2 \cdot 3^4 = 162$. Betty’s number has 9 distinct divisors and 2 and 3 are among them. By similar arguments, since $9 = 3 \cdot 3$ Betty’s number is $2^2 \cdot 3^2 = 36$. It is easy to check that for either Ann’s number, the number of distinct numbers on lists of both girls is 13.

9.3. Brothers found a treasure of gold and silver. They split it so that each share was 100kg. The oldest brother got $1/5$ of all gold, and $1/7$ of all silver. The youngest brother got $1/7$ of all gold. What part of all silver did the youngest brother get?

**Solution.** Let $x$ be amount of gold in the treasure. Then $100n - x$ is of amount of silver where $n$ is the number of brothers. Then the oldest brother got $\frac{1}{5}x + \frac{1}{7}(100n - x) = 100$ (kg) of the treasure and therefore $x = 1750 - 250n$. Then the total amount of silver is $350n - 1750$. Assuming that both amounts of gold and silver are positive we get $n = 6$. Therefore 250 and 350 are amounts of gold and silver in the treasure respectively. Hence the youngest brother got $250/7$ kg of gold and therefore $100 - 250/7 = 450/7$ of silver, which constitutes $1/7 \times 450/350 = 9/49$ of the total amount of silver.

9.4. A disc of radius 1 is given. Prove that one can cut out three pieces which could be rearranged into $1 \times 2.4$ rectangle. One can rotate and flip over pieces.

**Solution.** By simple calculations one can prove that it is possible to cut out of the given circle three pieces: $(1 \times \sqrt{3})$-rectangle and two $(\frac{1}{2}(\sqrt{3} - 1) \times 1)$ rectangles. These pieces can be rearranged into a new rectangle $(a \times 1)$ where $a = 2\sqrt{3} - 1 > 2.4$ as required.

9.5. Let $a$ and $n$ be natural numbers. Given that $a^n$ is 2014-digit number find the smallest $k$ such that $a$ cannot be a $k$-digit number.
SOLUTION. Let the number of digits of \( a \) be \( k \). It means that \( 10^{k-1} \leq a < 10^k \). Then \( 10^{(k-1)n} \leq a^n < 10^{kn} \). Hence for number of digits of \( a^n \) (which is 2014) we have boundaries \((k - 1)n + 1 \leq 2014 < nk + 1\) or

\[
\frac{2013}{k} < n \leq \frac{2013}{k - 1}
\]

(*)

If for some \( k \) there exists integer \( n \) which satisfies (*) then \( a \) is \( k \)-digit number. Therefore, we should look for \( k \) (and choose the smallest) for which there is no integer solution of (*). It is clear that if \( k \) grows, the difference \( \frac{2013}{k} - \frac{2013}{k - 1} \) gets smaller, so sooner or later we will find the first required \( k \).

First let us set \( \frac{2013}{k} - \frac{2013}{k - 1} < 1 \) (otherwise, there is a solution of (*)). Solving this inequality in natural numbers we get \( k \geq 46 \).

Next by brute force we check all \( k \) starting from 46: we have \( 2013/45 \approx 44.7 \), \( 2013/46 \approx 43.7 \), \( 2013/47 \approx 42.8 \), \( 2013/48 \approx 41.9 \) (so all corresponding intervals contain integers) and only \( 2013/49 \approx 41.08 \) and this interval does not contain an integer. We can see that the smallest \( k \) that works is 49.

9.6. Pavel invented a new way to add numbers. For two numbers \( a \) and \( b \) the pavelsum is defined as \( a \oplus b = (a + b)/(1 - ab) \) (if it is defined). He gave three numbers \( a, b \) and \( c \) to Boris and Michael and asked Boris to paveladd \( a \) and \( b \) and then paveladd \( c \) to the result while Michael was asked to paveladd \( b \) and \( c \), and then paveladd \( a \) to the result. Could Boris and Michael get different results?

SOLUTION. It is clear that \( a \oplus b = b \oplus a \) (commutative law). We can prove that \( (a \oplus b) \oplus c = a \oplus (b \oplus c) \) (associative law) by direct calculation. Another way to prove it is to plug \( a = \tan(\alpha), b = \tan(\beta) \) and \( c = \tan(\gamma) \) and notice that \( \tan(\alpha) \oplus \tan(\beta) = \tan(\alpha + \beta) \). Then

\[
(\tan(\alpha) \oplus \tan(\alpha)) \oplus \tan(\gamma) = \tan(\alpha + \beta) \oplus \tan(\gamma) = \tan(\alpha + \beta + \gamma).
\]

However there is question about the domain and it may happen that \( (a \oplus b) \oplus c \) is defined while \( a \oplus (b \oplus c) \) is not. Example: \( (a \oplus 1) \oplus 1 = -1/a \) provided \( a \neq 0, 1 \) but \( a \oplus (1 \oplus 1) \) is undefined since \( 1 \oplus 1 \) is undefined.
10.1. Given a convex pentagon. For each pair of its diagonals intersecting inside consider the smallest angle between them. Find all possible values of the sum of all these five angles.

Solution.

![Diagram of a convex pentagon with labeled vertices A, B, C, D, E, F, G, H, I, J.](image)

(a)

(b)

Solution. Let $\Sigma$ be the required sum. Consider convex pentagon $FGHIJ$ formed by diagonals. It is obvious that $\Sigma$ cannot be larger than the sum of adjacent angles of $FGHIJ$ which is $5 \cdot 180^\circ - S = 360^\circ$ where $S = 3 \cdot 180^\circ$ is the sum of inner angles. Therefore $\Sigma \leq 360^\circ$. If all inner angles of $FGHIJ$ are obtuse, $\Sigma = 360^\circ$.

Let us transform case (a) by moving points $E$ and $D$ far away to the left and $B$ and $C$ to the right (case (b)). We can see that $\angle JFG = \alpha$ and $\angle IHG = \beta$ can be made arbitrarily small. In comparison to case (a) the value of $\Sigma$ is decreased by the sum of $(180^\circ - 2\alpha)$ and $(180^\circ - 2\beta)$ so in (b) $\Sigma = 2(\alpha + \beta)$. Therefore it can be made arbitrarily small. Hence, the answer is $\Sigma \in (0^\circ, 360^\circ]$.

10.2. Let $f(x) = x^3 + 9x^2 + 27x + 24$. Solve equation $f(f(f(f(x)))) = 0$.

Solution. Observe that $f(x) = (x + 3)^3 - 3$. Then $f(f(x)) = ((x + 3)^3)^3 - 3 = (x + 3)^9 - 3$ and $f(f(f(f(x)))) = (x + 3)^{81} - 3$. Then the equation becomes $(x + 3)^{81} = 3$. Hence, $x = -3 + 3\sqrt[81]{3}$.

10.3. A disc of radius 1 is given. Prove that one can cut out four pieces which could be rearranged into $1 \times 2.5$ rectangle. One can rotate and flip over pieces.

Solution. By simple calculations one can prove that it is possible to cut out of the given circle four pieces: $(1 \times \sqrt{3})$-rectangle and two congruent pentagons, one of which we further split in two trapezoids each with height...
and bases $\frac{1}{2}$ and $\frac{1}{2}(\sqrt{3} - 1)$. These pieces can be rearranged into a new rectangle $(a \times 1)$ where $a = \sqrt{3} + \frac{1}{2}(\sqrt{3} - 1) + \frac{1}{2} = \frac{3}{2}\sqrt{3} > 2.5$ as required.

10.4. 100,000 squares were drawn inside a given square with the side 100. Diagonals of distinct inner squares do not intersect. Prove that at least one of the inner squares has a side length less than 1.

**Solution.** Let in each of 100,000 squares mark a center. Splitting the given square into 90,000 “tiles” (squares with the side $\frac{1}{3}$) we guarantee that two of the marked points (let it be $P$ and $Q$) belong the same tile (Pigeon Hole principle). Then the distance between $P$ and $Q$ is at most $r = \frac{\sqrt{2}}{3} < \frac{1}{\sqrt{2}}$.

Assume that there is no square with side less than 1. This implies that the half of diagonal (let us call it arm) of every square is at least $1/\sqrt{2} > r$. Consider two squares, $p$ and $q$ with centres at $P$ and $Q$ respectively. Note that $Q$ is inside of the circle with centre at $P$ and radius $1/\sqrt{2}$ (so, $PQ$ is smaller then the arm). From $Q$ drop two perpendiculars $QX$ and $QY$ to the arms of the square $p$. Then one of the arms of $q$ will be in a rectangle $YQXP$. Therefore its length will not exceed $PQ$ which means that the arms of squares $p$ and $q$ intersect. We got a contradiction due to a false assumption. Hence, at least one of the squares has the side less than 1.

10.5. Let $a$ and $n$ be natural numbers. Given that $a^n$ is 2014-digit number find the smallest $k$ such that $a$ cannot be a $k$-digit number.

**Solution.** Let the number of digits of $a$ be $k$. It means that $10^{k-1} \leq a < 10^k$. Therefore $10^{(k-1)n} \leq a^n < 10^{kn}$. Hence for number of digits of $a^n$
(which is 2014) we have boundaries \((k - 1)n + 1 \leq 2014 < nk + 1\) or

\[
\frac{2013}{k} < n \leq \frac{2013}{k-1}
\] (*)

If for some \(k\) there exists integer \(n\) which satisfies (*) then \(a\) is \(k\)-digit number. Therefore, we should look for \(k\) (and choose the smallest) for which there is no \(n\) integer solution of (*). It is clear that if \(k\) grows, the difference \(\frac{2013}{k-1} - \frac{2013}{k}\) gets smaller, so sooner or later we will find the first required \(k\).

Let us set \(\frac{2013}{k-1} - \frac{2013}{k} < 1\) (otherwise, there is a solution of (*)). Solving this inequality in natural numbers we get \(k \geq 46\).

By brute force we check all \(k\) starting from 46: we have \(2013/45 \approx 44.7\), \(2013/46 \approx 43.7\), \(2013/47 \approx 42.8\), \(2013/48 \approx 41.9\) (so all corresponding intervals contain integers) and only \(2013/49 \approx 41.08\) and this interval does not contain an integer. We can see that the smallest \(k\) that works is 49.

10.6. Pavel invented a new way to add numbers. For two numbers \(a\) and \(b\) the pavelsum is defined by \(a \oplus b = (a + b)/(1 - ab)\) (if it is defined). He gave four numbers \(a, b, c\) and \(d\) Boris and Michael and asked Boris to paveladd \(a\) and \(b\), and then paveladd \(c\), and finally paveladd \(d\) to the result while Michael was asked to paveladd \(c\) and \(d\), and then paveladd \(b\), and finally paveladd \(a\). Could Boris and Michael get different results?

**Solution.** It is clear that \(a \oplus b = b \oplus a\) (commutative law). We can prove that \(((a \oplus b) \oplus c) \oplus d = a \oplus (b \oplus (c \oplus d))\) by direct calculation. Another way to prove it is to plug \(a = \tan(\alpha), b = \tan(\beta), c = \tan(\gamma)\) and \(d = \tan(\delta)\) and notice that \(\tan(\alpha) \oplus \tan(\beta) = \tan(\alpha + \beta)\). Then

\[
(((\tan(\alpha) \oplus \tan(\alpha))) \oplus \tan(\gamma)) \oplus \tan(\delta) = \tan(\alpha + \beta + \gamma + \delta)
\]

However there is question about the domain and it may happen that the left-hand expression is defined while the right-hand expression is not. Example: \(((a \oplus 0) \oplus 1) \oplus 1 = -1/a\) provided \(a \neq 0, 1\) but \(a \oplus (0 \oplus (1 \oplus 1))\) is undefined since \(1 \oplus 1\) is undefined.
11.1. Given a convex pentagon. For each pair of its diagonals intersecting inside consider the smallest angle between them. Find all possible values of the sum of all these five angles.

Solution.

Given convex pentagon $FGHIJ$ formed by diagonals. It is obvious that $\Sigma$ cannot be larger than the sum of adjacent angles of $FGHIJ$ which is $5 \cdot 180^\circ - S = 360^\circ$ where $S = 3 \cdot 180^\circ$ is the sum of inner angles. Therefore $\Sigma \leq 360^\circ$. If all inner angles of $FGHIJ$ are obtuse, $\Sigma = 360^\circ$.

Let us transform case (a) by moving points $E$ and $D$ far away to the left and $B$ and $C$ to the right (case (b)). We can see that $\angle JFG = \alpha$ and $\angle IHG = \beta$ can be made arbitrarily small. In comparison to case (a) the value of $\Sigma$ is decreased by the sum of $(180^\circ - 2\alpha)$ and $(180^\circ - 2\beta)$ so in (b) $\Sigma = 2(\alpha + \beta)$. Therefore it can be made arbitrarily small. Hence, the answer is $\Sigma \in (0^\circ, 360^\circ]$.

11.2. Let $f(x) = x^3 + 9x^2 + 27x + 24$. Solve equation $f(f(f(f(x)))) = 0$.

Solution. Observe that $f(x) = (x + 3)^3 - 3$. Then $f(f(x)) = ((x + 3)^3 - 3 = (x + 3)^9 - 3$ and $f(f(f(f(x)))) = (x + 3)^{81} - 3$. Then the equation becomes $(x + 3)^{81} = 3$. Hence, $x = -3 + 3\sqrt{3}$.

11.3. A disc of radius 1 is given. Prove that one can cut out five pieces which could be rearranged into $1 \times 2.7$ rectangle. One can rotate and flip over pieces.

Solution. By simple calculations one can prove that it is possible to cut out of the given circle three pieces (look at the picture (a)): a central hexagon consisting of two identical trapezoids each with height $\frac{1}{2}$ and bases $\sqrt{3}$ and 2.
and two congruent pentagons, which we further split in two trapezoids each with height $\frac{1}{2}$ and bases $\sqrt{\frac{3}{2}} - \frac{1}{2}$ and $\frac{1}{2}$. These pieces can be rearranged into a new rectangle ($a \times 1$) where $a = \sqrt{3} + \frac{1}{2} + \frac{1}{2} = \sqrt{3} + 1 > 2.7$ as required.

11.4. Does there exist a tetrahedron with height of 60 cm, based perimeter 62 cm and the height of any lateral face (drawn to a side of the base) 61 cm?

**Solution.** Since each lateral face of the tetrahedron has the same height (drawn from vertex $S$), the height of the tetrahedron drops into the center of incircle of the base triangle. Then the radius of the incircle $r = \sqrt{61^2 - 60^2} = 11$. Therefore the area of the base is $\frac{1}{2}pr = 31 \cdot 11$. It is known that among triangles with the given perimeter $p$ an equilateral triangle has the greatest area. In our case we have $\frac{\sqrt{3}}{4}(p/3)^2 = \frac{\sqrt{3}}{9}31^2 \geq 31 \cdot 11$ which is false. Hence such tetrahedron does not exist.

11.5. Let $a$ and $n$ be natural numbers. Given that $a^n$ is 2014-digit number find the smallest $k$ such that $a$ cannot be a $k$-digit number.
Solution. Let the number of digits of \( a \) be \( k \). It means that \( 10^{k-1} \leq a < 10^k \). Then \( 10^{(k-1)n} \leq a^n < 10^{kn} \). Hence for number of digits of \( a^n \) (which is 2014) we have boundaries \((k - 1)n + 1 \leq 2014 < nk + 1\) or

\[
\frac{2013}{k} < n \leq \frac{2013}{k-1}
\]

(*)

If for some \( k \) there exists integer \( n \) which satisfies (*) then \( a \) is \( k \)-digit number. Therefore, we should look for \( k \) (and choose the smallest) for which there is no \( n \) integer solution of (*). It is clear that if \( k \) grows, the difference \( \frac{2013}{k-1} - \frac{2013}{k} \) gets smaller, so sooner or later we will find the first required \( k \).

First let us set \( \frac{2013}{k-1} - \frac{2013}{k} < 1 \) (otherwise, there is a solution of (*)). Solving this inequality in natural numbers we get \( k \geq 46 \).

Next by brute force we check all \( k \) starting from 46: we have \( 2013/45 \approx 44.7 \), \( 2013/46 \approx 43.7 \), \( 2013/47 \approx 42.8 \), \( 2013/48 \approx 41.9 \) (so all corresponding intervals contain integers) and only \( 2013/49 \approx 41.08 \) and this interval does not contain an integer. We can see that the smallest \( k \) that works is 49.

11.6. Pavel invented a new way to add numbers. For two numbers \( a \) and \( b \) the pavelsum is defined by \( a \oplus b = (a + b)/(1 - ab) \) (if it is defined properly). As usual he defined a pavelproduct of \( a \) and a natural number \( n \) as the pavelsum of \( n \) equal terms: \( a \otimes n = ((a \oplus a) \oplus a) \ldots \oplus a \). Does there exist two natural numbers \( x \neq y \) such that \( x \otimes y = y \otimes x \)?

Answer. No.

Solution. Unfortunately, author’s solution of the problem contains a crucial error.

Since \( \tan(\alpha) \oplus \tan(\beta) = \tan(\alpha + \beta) \) we conclude that \( x \oplus y = \tan(\arctan(x) + \arctan(y)) \) and therefore \( x \otimes y = \tan(y \arctan(x)) \) for positive integer \( y \) (proof by induction). Then \( x \otimes y = y \otimes x \) means that \( \tan(y \arctan(x)) = \tan(x \arctan(y)) \) and therefore

\[
y \arctan(x) = x \arctan(y) + \pi k.
\]

(*)

Therefore problem is to find if there exist natural numbers \( x \neq y \) and integer \( k \) satisfying (*). One can prove easily that for \( k = 0 \) such solution does not exist (indeed, calculating derivative one can prove that \( f(x) = \frac{\arctan(x)}{x} \) is a decaying function at \([0, \infty)\).

But it does not work for \( k \neq 0 \). Recently one of our colleagues sent a proof which is based on the properties of commutative ring \( \mathbb{Z}[i] \) of complex numbers with the integer both real and imaginary parts.