points problems

1. Pete has marked several (three or more) points in the plane such that all distances between them are different. A pair of marked points \( A; B \) will be called unusual if \( A \) is the furthest marked point from \( B \), and \( B \) is the nearest marked point to \( A \) (apart from \( A \) itself). What is the largest possible number of unusual pairs that Pete can obtain?

2. Given that \( 0 < a, b, c, d < 1 \) and \( abcd = (1 - a)(1 - b)(1 - c)(1 - d) \), prove that

\[
(a + b + c + d) - (a + c)(b + d) \geq 1.
\]

3. In triangle \( ABC \), points \( A_1, B_1, C_1 \) are bases of altitudes from vertices \( A, B, C \), and points \( C_A, C_B \) are the projections of \( C_1 \) to \( AC \) and \( BC \) respectively. Prove that line \( C_AC_B \) bisects the segments \( C_1A_1 \) and \( C_1B_1 \).

4. Does there exist a convex \( N \)-gon such that all its sides are equal and all vertices belong to the parabola \( y = x^2 \) for

3 a) \( N = 2011 \);
4 b) \( N = 2012 \)?

5. We will call a positive integer \( \text{good} \) if all its digits are nonzero. A good integer will be called \( \text{special} \) if it has at least \( k \) digits and their values strictly increase from left to right. Let a good integer be given. At each move, one may either add some special integer to its digital expression from the left or from the right, or insert a special integer between any two its digits, or remove a special number from its digital expression. What is the largest \( k \) such that any good integer can be turned into any other good integer by such moves?

6. Prove that the integer \( 1^1 + 3^3 + 5^5 + \ldots + (2^n - 1)^{2^{n-1}} \) is a multiple of \( 2^n \) but not a multiple of \( 2^{n+1} \).

7. 100 red points divide a blue circle into 100 arcs such that their lengths are all positive integers from 1 to 100 in an arbitrary order. Prove that there exist two perpendicular chords with red endpoints.