1. Let $O$ be the centre of the circle, $K$ be the point of tangency with $BC$ and $H$ be the point of intersection of $AC$ and $BD$. Since $AB = BC$, $AC$ is perpendicular to $OB$ by symmetry. Similarly, $BD$ is perpendicular to $OC$. Since $AC$ intersects $BD$ at $H$, $H$ is the orthocentre of triangle $OBC$. Now the radius $OK$ is perpendicular to the tangent $BC$. Hence the third altitude $OK$ of triangle $OBC$ passes through $H$.

2. Note that $b = a(10^n + 1)$ so that $\frac{b}{a^2} = \frac{10^n+1}{a}$. Suppose it is an integer $d$. Since $a$ is an $n$-digit number, $1 < d < 11$. Since $10^n + 1$ is not divisible by 2, 3 or 5, the only possible value for $d$ is 7. The example $a = 143$ and $b = 143143$ shows that we can indeed have $d = 7$.

3. Let the quadrilateral be $ABCD$ with $AC = 1001$ and $BD = n$. Note that $1002^2 - 1001^2 = 2003$ lies between $44^2$ and $45^2$. For $45 \leq n \leq 1001$, let $M$ be the common midpoint of $AC$ and $BD$. Initially, let $B$ lie on $AM$, so that the degenerate quadrilateral $ABCD$ has perimeter 2002. Now rotate $BD$ about $M$. When $BD$ is perpendicular to $AC$, the perimeter of $ABCD$ will exceed 2004. Hence at some point during the rotation, the perimeter of $ABCD$ is exactly 2004. It follows that all values of $n$ between 45 and 1001 inclusive are possible. For $2 \leq n \leq 44$, start with the rhombus $ABCD$ whose perimeter is less than 2004. Translate $BD$ in the direction $AC$. When $C$ is the midpoint of $BD$, both $AB$ and $AD$ are longer than 1001, so that the degenerate quadrilateral $ABCD$ has perimeter exceeding $2002 + n \geq 2004$. Hence at some point during the translation, the perimeter of $ABCD$ is exactly 2004. It follows that all values of $n$ between 2 and 44 inclusive are possible. Finally, consider the case $n = 1$. Let $M$ be the point of intersection of $AC$ and $BD$. Then

$$2004 = AB + BC + CD + DA$$
$$< MA + MB + MB + MC + MC + MD + MD + MA$$
$$= 2(AC + BD)$$
$$= 2004,$$

which is a contradiction. It follows that we cannot have $n = 1$.

4. Let the first three terms be $a_1 = a$, $a_2 = a + d$ and $a_3 = a + 2d$, where $d$ is the common difference. Let $a_2^2 = a + kd$, $a_2^2 = a + md$ and $a_3^2 = a + nd$ for some positive integers $k$, $m$ and $n$. Then $a^2 = a + kd$, $a^2 + 2ad + d^2 = a + md$ and $a^2 + 4ad + 4d^2 = a + nd$. It follows that $2ad + d^2 = nd - kd$ or $2a + d = m - k$, and $4ad + 4d^2 = nd - kd$ or $4a + 4d = n - k$. Eliminating $d$, we have $a = \frac{4m-n-3k}{4}$. Hence $a$ is an integer multiple of $\frac{1}{4}$. Eliminating $a$, we have $d = \frac{n+k-2m}{2}$. Hence $d$ is an integer multiple of $\frac{1}{2}$.

We consider the following cases.

(1) Let $\{a\} = 0$ and $\{d\} = \frac{1}{2}$. Every term of the progression is an integral multiple of $\frac{1}{2}$ but $a_2^2$ is not, a contradiction.

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1 Courtesy of Andy Liu.
(2) Let \( \{a\} = \frac{1}{2} \). Every term of the progression is an integral multiple of \( \frac{1}{2} \) but \( a_1^2 \) is not, a contradiction.

(3) Let \( \{a\} = \frac{1}{4} \) or \( \frac{3}{4} \). Every term of the progression is an integral multiple of \( \frac{1}{4} \) but \( a_1^2 \) is not, a contradiction.

Thus both \( a \) and \( d \) are integers, so that every term of the progression is an integer.

5. There are \( 9 \times 10^9 \) 10-digit numbers. If two of them are non-neighbours, they cannot have the same digits in each of the first nine places. Thus the number of 10-digit numbers we can choose is no more than the number of 9-digit numbers, which is \( 9 \times 10^8 \). On the other hand, for each 9-digit number, we can add a unique tenth digit so that the sum of all 10 digits is a multiple of 10. If two of the 10-digit numbers obtained this way differ in only one digit, not both digit sums can be multiples of 10. Hence no two are neighbours among these \( 9 \times 10^8 \) 10-digit numbers.