Solutions of
22nd Tournament of Towns
Advanced Level (Senior).

Problem 1.

Let us assume that \( a \geq b \geq c \geq d \). Since \( \text{lcm}(a, b, c, d) = a + b + c + d \), we have \( 4a \geq a + b + c + d = \{a, 2a, 3a, 4a\} \).

Case 1. \( a + b + c + d = a \) implies \( b + c + d = 0 \) which is impossible \((a, b, c, d \in \mathbb{N})\).

Case 2. \( a + b + c + d = 4a \) implies that \( a = b = c = d \) and \( \text{lcm}(a, b, c, d) = a \) but not \( 4a \).

Case 3. \( a + b + c + d = 3a \) implies that \( \text{lcm}(a, b, c, d) \) is divisible by 3. Therefore \( abcd \) is divisible as well.

Case 4. \( a + b + c + d = 2a \) or \( a = b + c + d \).

In this case we have: \( 6b \geq \text{lcm}(a, b, c, d) = 2(b + c + d) = \{b, 2b, 3b, 4b, 5b, 6b\} \).

(4a) \( \text{lcm}(a, b, c, d) = \{b, 2b\} \) are impossible.

(4b) \( \text{lcm}(a, b, c, d) = \{3b, 5b, 6b\} \) imply that \( abcd \) is divisible by 5 or 3.

(4c) \( \text{lcm}(a, b, c, d) = 4b \). In this case we have \( 2(b + c + d) = 4b \).

This means that \( b = c + d \) and \( a = 2(c + d) \).

Then \( 8c \geq \text{lcm}(a, b, c, d) = a + b + c + d = 4(c + d) = \{c, 2c, 3c, 4c, 5c, 6c, 7c, 8c\} \).

The only relevant cases to consider are: \( \text{lcm}(a, b, c, d) = \{5c, 6c\} \), \( abcd \) is divisible by 5 or 3. In case of \( \text{lcm}(a, b, c, d) = 8c \), we have \( 4(c+d)=8c \); so \( d = c, b = 2c \) and \( a=4c \). Therefore, \( \text{lcm}(a, b, c, d) = 4c \) but not \( 8c \). In the last case when \( \text{lcm}(a, b, c, d) = 7c \), we have: \( 4(c + d) = 7c \) or \( 3c = 4d \) which implies that 3 divides \( d \) and therefore \( abcd \) is divisible by 3.

All cases were considered.

Problem 2.

Let us place a cube with side of 1 unit in a coordinate system as shown on the picture.

It is easy to see that a plane passing through points \( P(0, 0, \frac{3}{2}) \), \( R(0, \frac{3}{2}, 0) \), \( Q(\frac{3}{2}, 0, 0) \) intersects the edges of the cube at midpoints \( M, N, K, L, I, J \) creating a regular hexagon (all sides of the hexagon are equal and all main diagonals are equal as well).

Now let us cut congruent isosceles triangles from each corner of the hexagon so that the base of each triangle is equal to the segment of the hexagon between two triangles.

It is easy to see that the constructed dodecagon is regular.

Let us prove that \( n \) cannot exceed 12. Really, a cube has 6 faces and we cannot take more than two points on each face of the cube since a polygon is still a dodecagon (in the case of collinear points) or a plane to which the points belong contains a face.
Problem 3. Given that
\[ AB = c, \quad AC = b, \quad BC = a, \quad BB'' = CC'' = a, \quad BB' = AA' = c \quad (c > b > a), \]
we have to find \( \frac{A'B'}{B''C''} \).

Let \( R \) be an intersection of \( A'B' \) and extension of \( B''C \).
First, let us note that \( B''C \parallel AB' \) (\( \triangle ABB' \) is isosceles and \( BC = a = BB'' \)). It implies that

1. \( \angle B'AC = \angle ACB'' = \angle RCA' \)
2. \( \frac{AB'}{B''C} = \frac{BB'}{BC} = \frac{AA'}{CC''} \).

Second, \( CR \parallel AB' \) implies that

3. \( \triangle A'CR \cong \triangle AA'B' \quad \Rightarrow \)
4. \( \frac{CR}{AB'} = \frac{CA'}{AA'} \).

It follows from (2),(4) that \( \frac{CR}{AC} = \frac{CB''}{CC''} \). This and (1) imply that \( \triangle B''CC'' \cong \triangle A'CR \). Finally, taking (3) into consideration, we have

\[ \frac{A'B'}{B''C''} = \frac{AA'}{CC''} = \frac{c}{a}. \]

Problem 4.
Let us consider a linear fraction \( f(x) = \frac{\alpha x + \beta}{\gamma x + \delta} \) and associate a matrix \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) with it. Note that two linear fractions are equal if and only if their associated matrices are proportional. Then \( a + \frac{1}{f(x)} = \frac{(\gamma + a\alpha)x + (\delta + a\beta)}{\alpha x + \beta} \) has an associated matrix

\[ \begin{pmatrix} \gamma + a\alpha & \delta + a\beta \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . \]

Note that a unit matrix is associated with \( f(x) = x \).
We can consider the given identity as a matrix equation

\[(1) \quad T_1 T_2 \cdots T_n I = \lambda I\]

where \(T_k = \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}\).

**Part a.** Let us use the theorem that the determinant of the product of the matrices is equal to the product of their determinants. Since the determinant of \(T_k\) equals \(-1\) and the determinant of \(\lambda I\) is equal to \(\lambda^2\), (1) implies that \((-1)^n = \lambda^2\) which is impossible for odd \(n\).

**Part b.** Note that in case \(n = 2\)

\[\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + 1 & a_1 \\ a_2 & 1 \end{pmatrix}\]

and (1) holds if and only if \(a_1 = a_2 = 0\). So \(n = 2\) is impossible.

However, for \(n = 4\) one can check that \(a_1 = a_3 = 1, a_2 = a_4 = -2\) solve the problem.

**Problem 5.**

Let us consider \(m \times n\)-table consisting of black and white squares. Let \(k\) be the largest difference between the number of black and white squares in any column or row. Let us assume that \(k > 0\). We can assume that this maximum is achieved in a column (we call it the main column) and that the number of black squares is larger than the number of white squares. For all the columns and rows we have

\[(1) \quad |B_i - W_i| \leq k\]

where \(B_i\) and \(W_i\) are the numbers of black and white squares.

If a rook is on any black square of the main column, it has to attack at least \(k\) more white than black squares in the row containing the chosen black square. This implies that in every such row we have \(W_i - B_i = k\). Let us call a row with this property the main row.

Let us consider any white square in the main row. Applying the previous arguments we can see that the difference between the number of black and white squares in the column containing this white square is exactly \(k\).

One can see that there are \(\frac{m+k}{2}\) black and \(\frac{m-k}{2}\) white squares in the main column and \(\frac{n+k}{2}\) white and \(\frac{n-k}{2}\) black squares in the main row. The number of main rows is at least \(\frac{n+k}{2}\) and the number of main columns is at least \(\frac{n+k}{2}\).

Now let us estimate the number of white squares in the table. The number of white squares in all main rows is \(\frac{n+k}{2} \cdot \frac{m+k}{2}\). For all other rows from (1) we have \(W_i \geq B_i - k\) which implies that \(W_i \geq \frac{n-k}{2}\). So, the number of white squares in all (not main) rows is not less than \(\frac{n-k}{2} \cdot \frac{m-k}{2}\). Therefore the total number of white squares in the table is not less than

\[\frac{n+k}{2} \cdot \frac{m+k}{2} + \frac{n-k}{2} \cdot \frac{m-k}{2} = \frac{mn + k^2}{2}\]

which is more than half of all squares in the table.

Applying the same argument to estimate the number of black squares starting with the main row, we get that the number of black squares is more than half of all squares in the table.

This contradiction proves that \(k = 0\) and therefore every column or row has an equal number of black and white squares.
Problem 6.

Part (a). Let us consider a square $ABCD$ with side $a = 1$ covering a nail with a radius $r = 0.05$ and the center $O$. Let $LM$ be a segment of the length $\ell$ lying on the border of the square. Let $\theta$ be an angle, under which we see $LM$ from $O$. One can prove that the smallest possible value of $\theta$ is achieved when $O$ lies on the distances $r$ from sides $AB$ and $BC$ and $A = L$.

From triangle $AMO$ we have $\frac{\ell}{\sin \theta} = \frac{OM}{\sin \angle MAO}$ and since $\sin \theta < \theta$ and $\sin \angle MAO = \frac{r}{OA} > \frac{r}{a}$ we conclude that $\frac{\ell}{\theta} < \frac{a^2}{r} = 20$.

Note that the union $F$ of several squares covering a nail is a star-like region: any ray coming from $O$ intersects the boundary of $F$ only once. Since the boundary of $F$ consists of described segments and the total angle is $2\pi$ we conclude that the perimeter $P$ of $F$ does not exceed $2\pi \cdot 20 < 130$.

Part (c). Let us nail into a board non-overlapping nails with a radius $r = 0.05$ densely enough to be sure that any square completely covers at least one nail. Then the perimeter $P$ of our figure does not exceed the sum of the perimeters $P_i$ of the figures obtained by squares covering $i$-th nail. Due to part (a), $P < 40\pi N$ where $N$ is the number of covered nails. But area $S$ of all squares is no less than the total area of the nails which is $\pi r^2 N$ and therefore $\frac{P}{S} < 16,000$.

Let us note that by increasing $r$ we can get a better estimate.

Part (b) follows from part (c).