Some Extra Credit Problems: Due 10/30/2013

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Problems

Complete any or as many of the following problems as you like and hand in carefully written solutions at the beginning of class on Wednesday, October 30. The problems have varying degree of difficulty, some being quite hard. Each fully correct problem accounts for an extra 5 points on your midterm score (i.e. 1% of your course grade overall).

1. (a) Prove that
\[ |1 + z + \cdots z^N| \leq \frac{2}{1-|z|} \]
for all \( z \in \mathbb{D} \). Notice the RHS is independent of \( N \).

(b) Prove that
\[ \left| \sum_{k=0}^{N} \frac{z^{2k}}{2 + z^k + z^{5k}} \right| \leq \frac{1 - |z|^{2N+2}}{2(1 - |z|)^2} \]
for \( z \in \mathbb{D} \).

2. Let \( z_1, \ldots, z_n \) be complex numbers in \( \mathbb{D} \). Prove that there exist numbers \( \epsilon_k = \pm 1 \), \( k = 1, \ldots, n \) such that
\[ |\sum_{k=1}^{m} \epsilon_k z_k| \leq \sqrt{3} \quad m = 1, \ldots, n \]

3. Show that four points \( z_1, z_2, z_3, z_4 \in \mathbb{C} \) are on the same complex circle (or extended line) if and only if the number
\[ \frac{\overline{z_1-z_2}}{\overline{z_1-z_3}} \frac{\overline{z_2-z_4}}{\overline{z_3-z_4}} \]
is real-valued.

4. Give an example of a map \( t \mapsto z(t) \) from an interval \( t \in (a, b) \subset \mathbb{R} \) into \( \mathbb{C} \) such that \( \lim_{t \to b} z(t) \) does not exist but \( \lim_{t \to b} \Re(z(t)) \) does.
5. Show that the series
\[ \sum_{n=1}^{\infty} \left( \frac{1}{z-n} + \frac{1}{n} \right) \]
converges for every \( z \notin \mathbb{N} \). Show that the convergence is in fact uniform on any compact set not intersecting \( \mathbb{N} \).

6. Suppose \( f \) is holomorphic on \( \mathbb{D} \) and that its range is a proper subset of \( \mathbb{D} \). Let \( a \in \mathbb{D}/f(\mathbb{D}) \). Is there a holomorphic \( F(z) \) satisfying
\[ e^{F(z)} = \frac{f(z) - a}{1 - \bar{a}f(z)} \]
and, if there is what is its derivative? Moreover, regardless of holomorphy, show that \( \Re(F(z)) < 0 \) for any \( F \) satisfying the above.

7. For \( l = 1, \ldots, N \) let \( I_l = [\alpha_l, \beta_l] \) with \( I_l \cap I_k = \emptyset \) for \( k \neq l \). Show that the function
\[ f(z) = \prod_{l=1}^{N} \left( \frac{z - \alpha_l}{z - \beta_l} \right) \]
has a holomorphic square root on \( \mathbb{C}/\bigcup_{l=1}^{N} I_l \).

8. Let \( \alpha \in \mathbb{C} \) and define
\[ f_\alpha(z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!} z^n \]
Show that when \( \alpha \in \mathbb{N} \), \( f_\alpha(z) \) is a polynomial. Show that for any \( \alpha \in \mathbb{C} \) \( f_\alpha \) is holomorphic on the unit disk \( \mathbb{D} \). Also, establish the following identities
\[ f'_\alpha(z) = \frac{\alpha f_\alpha(z)}{1 + z} \]
\[ f_{\alpha+\beta}(z) = f_\alpha(z)f_\beta(z), \quad |z| < 1 \]