MAT 445/1196 - Complex symplectic Lie algebras

Let \( n \) be an integer greater than or equal to 2. Let \( J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \).

Then
\[
Sp_{2n}(\mathbb{C}) = \{ g \in GL_{2n}(\mathbb{C}) \mid {}^t g J g = J \} \\
sp_{2n}(\mathbb{C}) = \{ X \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid {}^t X J + J X = 0 \}.
\]

Or, define a nondegenerate bilinear symplectic form on \( \mathbb{C}^{2n} \) by \( Q(x, y) = {}^t x J y \).

Then
\[
Sp_{2n}(\mathbb{C}) = \{ g \in GL_{2n}(\mathbb{C}) \mid Q(g x, g y) = Q(x, y), \forall x, y \in \mathbb{C}^{2n} \} \\
sp_{2n}(\mathbb{C}) = \{ X \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid Q(X x, y) + Q(x, X y) = 0, \forall x, y \in \mathbb{C}^{2n} \}.
\]

We can write elements of \( \mathfrak{g} = sp_{2n}(\mathbb{C}) \) in block form: \( X = \begin{pmatrix} A & B \\ C & -{}^t A \end{pmatrix} \),
where \( A, B, C \in M_{n \times n}(\mathbb{C}) \) and \( B = {}^t B, C = {}^t C \). Note that the dimension of \( sp_{2n}(\mathbb{C}) \) is \( n(2n + 1) \).

The set of diagonal matrices \( \mathfrak{h} \) in \( \mathfrak{g} = sp_{2n}(\mathbb{C}) \) is an abelian subalgebra of \( \mathfrak{g} \). The elements \( H_i = E_{i,i} - E_{n+i,n+i}, 1 \leq i \leq n \), form a basis of the vector space \( \mathfrak{h} \). The subalgebra \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \).

Let \( \{ \lambda_1, \ldots, \lambda_n \} \) be the basis of \( \mathfrak{h}^* \) that is dual to the basis \( \{ H_1, \ldots, H_n \} \) of \( \mathfrak{h} \): that is, \( \lambda_j(H_i) = \delta_{ij}, 1 \leq i, j \leq n \).

Consider the adjoint representation \( X \mapsto \text{ad} X \) of \( \mathfrak{g} \): \( \text{ad} X : \mathfrak{g} \to \mathfrak{g} \) is defined by \( \text{ad} X(Y) = [X, Y], Y \in \mathfrak{g} \). The set \( \text{ad} \mathfrak{h} = \{ \text{ad} H \mid H \in \mathfrak{h} \} \) is a commuting family of semisimple endomorphisms of \( \mathfrak{g} \). Hence the operators in \( \text{ad} \mathfrak{h} \) are simultaneously diagonalizable. There exists a finite set \( \Phi = \Phi(\mathfrak{g}, \mathfrak{h}) \) of nonzero elements of \( \mathfrak{h}^* \) such that
\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \text{ where } \mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid \text{ad} H(X) = [H, X] = \alpha(H)X \forall H \in \mathfrak{h} \}.
\]

This is called the Cartan decomposition of \( \mathfrak{g} \). Any semisimple Lie algebra has an analogous Cartan decomposition, relative to the restriction of the adjoint representation of \( \mathfrak{g} \) to a Cartan subalgebra of \( \mathfrak{g} \). The given Cartan subalgebra \( \mathfrak{h} \) is always equal to the space \( \{ X \in \mathfrak{g} \mid [H, X] = 0 \forall H \in \mathfrak{h} \} \). The elements of \( \Phi \) are called the roots of \( \mathfrak{g} \) (relative to \( \mathfrak{h} \)). If \( \alpha \in \Phi \), the subspace \( \mathfrak{g}_\alpha \) is one-dimensional and is called the root space corresponding to \( \alpha \).
Root spaces for $sp_{2n}(\mathbb{C})$:
If $1 \leq i \neq j \leq n$, then $X_{ij} := E_{i,j} - E_{n+j,n+i}$ spans $g_{\lambda_i - \lambda_j}$.
If $1 \leq i < j \leq n$, then $Y_{ij} := E_{i,n+j} + E_{j,n+i}$ spans $g_{\lambda_i + \lambda_j}$.
If $1 \leq i < j \leq n$, then $Z_{ij} := E_{n+i,j} + E_{n+j,i}$ spans $g_{-\lambda_i - \lambda_j}$.
If $1 \leq i \leq n$, then $U_i := E_{i,n+i}$ spans $g_{2\lambda_i}$.
If $1 \leq i \leq n$, then $V_i := E_{n+i,i}$ spans $g_{-2\lambda_i}$.

Hence the roots for $sp_{2n}(\mathbb{C})$ are
\[ \Phi = \{ \pm (\lambda_i - \lambda_j), \pm (\lambda_i + \lambda_j), 1 \leq i < j \leq n; \pm 2\lambda_i, 1 \leq i \leq n \} \]

In the case of $sp_4(\mathbb{C})$, we have $H_1 = \text{diag}(1, 0, -1, 0), H_2 = \text{diag}(0, 1, 0, -1),$

\[
X_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},
X_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
Y_{12} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
Z_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
U_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
V_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.

Let $\alpha, \beta \in \Phi$. Let $X_\alpha \in g_\alpha$ and $X_\beta \in g_\beta$. The element $[X_\alpha, X_\beta]$, being an element of $g$ has a decomposition as a sum of an element in $h$ and some elements in various root spaces. To determine this decomposition, we evaluate $[H, [X_\alpha, X_\beta]]$ for $H \in h$. The Jacobi identity tells us that
\[ [H, [X_\alpha, X_\beta]] + [X_\alpha, [X_\beta, H]] + [X_\beta, [H, X_\alpha]] = 0. \]

Since $[X_\beta, H] = -[H, X_\beta] = -\beta(H)X_\beta$ and $[H, X_\alpha] = \alpha(H)X_\alpha$, this can be rewritten to get
\[ [H, [X_\alpha, X_\beta]] = \beta(H)[X_\alpha, X_\beta] - \alpha(H)[X_\beta, X_\alpha] = (\alpha + \beta)(H)[X_\alpha, X_\beta]. \]
It follows that

\[
[X_\alpha, X_\beta] \in \begin{cases} 
\mathfrak{h}, & \text{if } \beta = -\alpha, \\
\mathfrak{g}_{\alpha+\beta}, & \text{if } \alpha + \beta \in \Phi, \\
\{0\}, & \text{otherwise.}
\end{cases}
\]

We can see that in the example \( g = \mathfrak{sp}_{2n}(\mathbb{C}) \), we have \( \alpha \in \Phi \) if and only if \( -\alpha \in \Phi \). This is true in general.

There are certain distinguished subalgebras \( \mathfrak{s}_\alpha \) of \( g \) attached to elements \( \alpha \) of \( \Phi \). Each of these subalgebras is isomorphic to \( \mathfrak{sl}_2(\mathbb{C}) \). Consider the root \( \alpha = \lambda_1 - \lambda_2 \) of \( \mathfrak{sp}_4(\mathbb{C}) \). Note that \( X_{12} \in \mathfrak{g}_\alpha, X_{21} \in \mathfrak{g}_{-\alpha} \), and \( [X_{12}, X_{21}] = \text{diag}(1, -1, -1, 1) = H_1 - H_2 \). We can easily see that the subspace \( \mathfrak{s}_\alpha := \text{Span}\{ H_1 - H_2, X_{12}, X_{21} \} \) is a subalgebra of \( \mathfrak{sp}_4(\mathbb{C}) \) that is isomorphic to \( \mathfrak{sl}_2(\mathbb{C}) \). In fact, if we let \( H_\alpha \) be the unique element of the one-dimensional subspace \( [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \) of \( \mathfrak{h} \) such that \( \alpha(H_\alpha) = 2 \), fixing a nonzero element \( X_\alpha \in \mathfrak{g}_\alpha \), we can find a nonzero element \( Y_\alpha \in \mathfrak{g}_{-\alpha} \) such that \( [X_\alpha, Y_\alpha] = H_\alpha \). With these choices, \( H_\alpha \mapsto \text{diag}(1, -1), X_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( Y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) extends to a Lie algebra isomorphism between \( \mathfrak{s}_\alpha \) and \( \mathfrak{sl}_2(\mathbb{C}) \).

If \( g = \mathfrak{sp}_4(\mathbb{C}) \), set \( \alpha = \lambda_1 - \lambda_2 \) and \( \beta = 2\lambda_2 \). Then

\[ \Phi = \{ \pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta) \} \]

With this labelling we have

\[
\begin{align*}
H_\alpha &= H_1 - H_2 = \text{diag}(1, -1, -1, 1) \\
H_\beta &= H_2 = \text{diag}(0, 1, 0, -1) \\
H_{\alpha+\beta} &= H_1 + H_2 = \text{diag}(1, 1, -1, -1) \\
H_{2\alpha+\beta} &= H_1.
\end{align*}
\]

It is immediate from the definition that \( H_{-\gamma} = -H_\gamma \) for \( \gamma \in \Phi \). When referring to the case \( g = \mathfrak{sp}_4(\mathbb{C}) \), we will reserve the notation \( \alpha \) for \( \lambda_1 - \lambda_2 \). However, when referring to general complex semisimple Lie algebras, \( \alpha \) will simply denote any element of \( \Phi \).

Let \( \Lambda_\Phi \) be the subset of \( \mathfrak{h}^* \) made up of all integral linear combinations of elements of \( \Phi \), and let \( \Lambda_W = \{ \lambda \in \mathfrak{h}^* \mid \lambda(H_\alpha) \in \mathbb{Z}, \forall \alpha \in \Phi \} \).

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Given a root $\alpha \in \Phi$, let $w_\alpha$ be the involution of the vector space $\mathfrak{h}^*$ defined as follows: $w_\alpha(\alpha) = -\alpha$, and $w_\alpha(\lambda) = \lambda$ for all $\lambda \in \Omega_\alpha := \{ \lambda \in \mathfrak{h}^* \mid \lambda(H_\alpha) = 0 \}$. Then

$$w_\alpha(\lambda) = \lambda - (2\lambda(H_\alpha)/\alpha(H_\alpha))\alpha = \lambda - \lambda(H_\alpha)\alpha, \quad \lambda \in \mathfrak{h}^*.$$  

The Weyl group of $\mathfrak{g}$ (or of $\Phi$) is defined to be the subgroup of $GL(\mathfrak{h}^*)$ generated by the set $\{ w_\alpha \mid \alpha \in \Phi \}$. Note that it is immediate from the definitions that $w_\alpha = w_{-\alpha}$ for all $\alpha \in \Phi$.

In the case of $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$, the set $\{ \alpha, \beta \}$ is a basis of $\mathfrak{h}^*$ and

$$\Omega_\alpha = \{ \lambda \in \mathfrak{h}^* \mid \lambda(H_1) = \lambda(H_2) \} = \text{Span}\{ \alpha + \beta \}$$

$$\Omega_\beta = \{ \lambda \in \mathfrak{h}^* \mid \lambda(H_2) = 0 \} = \text{Span}\{ 2\alpha + \beta \}$$

$$w_\alpha(\beta) = 2\alpha + \beta, \quad w_\beta(\alpha) = \alpha + \beta$$

It is easy to check that $w_\alpha w_\beta$ has order 4, $W$ is generated by $\{ w_\alpha, w_\beta \}$, and $W$ is isomorphic to the dihedral group of order 8. Furthermore, $W(\Phi) = \Phi$ - this is true in general.

It is possible to show that every element of $W$ is induced by an automorphism of $\mathfrak{g}$ that carries $\mathfrak{h}$ to itself. In fact, if $G$ is a complex Lie group with Lie algebra $\mathfrak{g}$, and $T$ is the Cartan subgroup of $G$ that corresponds to $\mathfrak{h}$ (that is, $T$ is the closed subgroup of $G$ that is generated by the exponentials of the elements of $\mathfrak{h}$), the group $W$ can be realized as $N_G(T)/T$, where $N_G(T)$ is the normalizer of $T$ in $G$. Given $\alpha \in \Phi$, there is an element $g_\alpha \in N_G(T)$ such that conjugation by $g_\alpha$ induces an automorphism $\text{Ad}g_\alpha$ of $\mathfrak{g}$ that preserves $\mathfrak{h}$ and restricts to an automorphism of $\mathfrak{h}$ that corresponds to the automorphism $w_\alpha$ of $\mathfrak{h}^*$.

In the example $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$, $T$ is the group of diagonal matrices in $Sp_4(\mathbb{C})$. For the given choice of $\alpha = \lambda_1 - \lambda_2$, we can take

$$g_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and $\text{Ad}g_\alpha(X) = g_\alpha X g_\alpha^{-1}, X \in \mathfrak{sp}_4(\mathbb{C})$. Restricting to $\mathfrak{h}$ and then composing, we obtain the involution $w_\alpha$ of $\mathfrak{h}^*$.

Up to multiplication by scalars, there is a unique inner product on $\mathfrak{h}^*$ that is $W$-invariant. For $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$, since $W$ is generated by $w_\alpha$ and $w_\beta$, it suffices
to take an inner product on $\mathfrak{h}^*$ that is $w_\alpha$ and $w_\beta$-invariant. Denoting the inner product by $\langle \cdot, \cdot \rangle$, we must have $\alpha$ orthogonal to $\Omega_\alpha$, that is, $\langle \alpha, \alpha + \beta \rangle = 0$, and $\beta$ orthogonal to $\Omega_\beta$. Hence

$$\langle \alpha, \beta \rangle = -\langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle / 2.$$  

We can (and do) normalize so that $\alpha$ is a unit vector. For convenience, we fix an isometry between our inner product space $\mathfrak{h}^*$ and with the inner product space $\mathbb{C}^2$ (relative to the standard inner product), with $\alpha$ identified with $(1, 0)$. In that case, there are two possible choices for the vector that we identify with $\beta$: $(-1, \pm 1)$. We choose to take $(-1, 1)$. Then $\alpha + \beta$ is identified with $(0, 1)$ and $2\alpha + \beta$ with $(1, 1)$.

It is convenient to partition $\Phi$ as a disjoint union of two sets $\Phi^+$ and $\Phi^-$ in a nice way. One of the properties we need from such a partition is: $\alpha \in \Phi^+$ if and only if $-\alpha \in \Phi^-$. Also, if $\alpha$ and $\beta$ belong to $\Phi^+$, we require that if $\alpha + \beta$ belongs to $\Phi$, it belongs to $\Phi^+$.

For example, we can choose a (real) linear functional $\ell$ on the space $\text{Span}_\mathbb{R}(\Phi)$ that is nonvanishing on the subset $\Phi$. Then we can set $\Phi^+ = \{ \alpha \in \Phi \mid \ell(\alpha) > 0 \}$ and $\Phi^- = \{ \alpha \in \Phi \mid \ell(\alpha) < 0 \}$. The elements of $\Phi^+$ are referred to as positive roots, and the elements of $\Phi^-$ are negative roots. A choice of $\Phi^+$ (and hence $\Phi^-$) is called an ordering on $\Phi$.

In our $\mathfrak{sp}_4(\mathbb{C})$ example, one possible choice for $\Phi^+$ is $\Phi^+ = \{ \alpha, \beta, \alpha + \beta, 2\alpha + \beta \}$.

A positive root is said to be simple (or primitive) if it cannot be expressed as a sum of two positive roots. (A similar definition can be made for negative roots). For the choice of $\Phi^+$ that we have made for $\mathfrak{sp}_4(\mathbb{C})$, $\alpha$ and $\beta$ are the simple roots in $\Phi^+$.

In general, suppose that $\Delta = \{ \alpha_1, \ldots, \alpha_\ell \}$ is the set of simple roots in $\Phi^+$. Then $\Delta$ is a basis for $\mathfrak{h}^*$ and

$$\Phi^+ \subset \left\{ \sum_{i=1}^\ell m_i \alpha_i \mid m_i \in \mathbb{Z}, m_i \geq 0 \right\}.$$  

To be continued....