MAT 240F - Problem Set 7

Due Thursday, November 20th

Questions 1, 3 b), 4 c) and 5 will be marked.

1. Let $V_1, V_2, W_1$ and $W_2$ be vector spaces over a field $F$. Let $T \in \mathcal{L}(V_1, V_2)$, $U_1 \in \mathcal{L}(W_1, V_1)$ and $U_2 \in \mathcal{L}(V_2, W_2)$. Suppose that $\text{nullity}(T)$ is finite and $U_1$ and $U_2$ are isomorphisms. (For full marks, do not assume that the vector spaces $V_j$ and $W_j$ are finite-dimensional.) Prove that $\text{nullity}(U_2TU_1) = \text{nullity}(T)$.

2. Let $F$ be a field and let $n$ be an integer such that $n \geq 2$. Suppose that $A, B \in M_{n \times n}(F)$. We say that $A$ is similar to $B$ if there exists an invertible matrix $C \in M_{n \times n}(F)$ such that $A = C^{-1}BC$.
   
   a) Show that if $A$ is similar to $B$, then $\text{rank}(A) = \text{rank}(B)$.
   
   b) Show that if $A$ is similar to $B$, then $A$ is invertible if and only if $B$ is invertible.
   
   c) Suppose that $A$ and $B$ are invertible. Show that $A$ is similar to $B$ if and only if $A^{-1}$ is similar to $B^{-1}$.
   
   d) Show that if $A$ is similar to $B$, then $A^m$ is similar to $B^m$ for all positive integers $m$.

3. For each $T \in \mathcal{L}(V, W)$ as defined below, find $T^{-1}(y)$ for each vector $y \in W$.
   
   a) Let $V = W = P_2(\mathbb{C})$ and let $(Tf)(x) = f(ix) - f(x-1) + f(0)$, $f \in P_2(\mathbb{C})$.
   
   b) Let $V = P_3(\mathbb{R})$ and $W = M_{2 \times 2}(\mathbb{R})$, and let
   
   $$T(ax^3 + bx^2 + cx + d) = \begin{pmatrix} a + c & \cdots & b \\ c & \cdots & a \\ \vdots & \ddots & \vdots \\ b & \cdots & c \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$
   
   c) Let $V = W = F_5^3$ and
   
   $$T(a, b, c) = ((2a + b + 3c) \mod 5, (a + b) \mod 5, (a + b + c) \mod 5), \quad a, b, c \in F_5.$$

4. Compute $\text{rank}(T)$ for each linear transformation $T$. Explain your answer fully.
   
   a) Let $T : M_{2 \times 2}(\mathbb{C}) \rightarrow \mathbb{C}^4$ be defined by $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a + i c + d, 2i a - b + i d, a - 3 c, b - (1 + 3i)c + (1 - i)d)$, $a, b, c$ and $d \in \mathbb{C}$.
   
   b) Let $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ be defined by $T(a + bx + cx^2) = a + 2b + c + (a + 3b + 4c)x + (2a + 3b - c)x^2$, $a, b, c \in \mathbb{R}$.
   
   c) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^5$ be defined by $T(a, b, c, d) = (4a - b + c, b + 3d, 3a + c + d, c - d, b + c + 2d)$, $a, b, c, d \in \mathbb{R}$.

5. Suppose that $A, B \in M_{n \times n}(F)$ are invertible. Prove that it is possible to transform $A$ into $B$ using elementary row operations. (Note: Elementary column operations should not be used.)

6. Suppose that $A, B \in M_{m \times n}(F)$, and $\text{rank}(A) = \text{rank}(B)$. Prove that there exist invertible matrices $P \in M_{m \times m}(F)$ and $Q \in M_{n \times n}(F)$ such that $B = PAQ$.

7. #14, §3.2.

8. #21, §3.2.