Mat 1197 - Representations of the mirabolic subgroup
February 9

Let
\[
H = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \mid a \in F^\times, x \in F \right\},
\]
\[
N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in F \right\},
\]
\[
S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in F^\times \right\}.
\]

The group \( H \) is called the mirabolic subgroup of \( GL_2(F) \). Let \( (\pi, V) \) be a smooth representation of \( N \) and let \( \vartheta \) be a quasicharacter of \( N \) (that is, a one-dimensional smooth representation of \( N \)). Let
\[
V(\vartheta) = \text{Span}(\{ \pi(n)v - \vartheta(n)v \mid n \in N, v \in V \}) \quad \text{and} \quad V_\vartheta = V/V(\vartheta).
\]

When \( \vartheta \) is trivial, we write \( V(N) \) instead of \( V(\vartheta) \) and \( V_N \) instead of \( V_\vartheta \).

1. Let \( (\pi, V) \) be a smooth representation of \( N \) and let \( v \in V \). Show that \( v \in V(\vartheta) \) if and only if there exists a compact open subgroup \( U \) of \( N \) such that \( \int_U \vartheta(n)^{-1}\pi(n)v \, dn = 0 \). (Here, \( dn \) is a Haar measure on \( N \).)

2. If \( (\pi_j, V_j) \) are smooth representations of \( N \), \( 1 \leq j \leq 3 \) and \( V_1 \to V_2 \to V_3 \) an exact sequence of \( N \)-morphisms, show that there is a corresponding exact sequence at the level of the spaces \( (V_j)_\vartheta \).

3. Suppose that \( \vartheta \) is nontrivial. Show that the inclusion \( V(N) \to V \) induces an isomorphism \( V(N)_\vartheta \simeq V_\vartheta \).

4. Let \( (\pi, V) \) be a smooth representation of \( N \). Prove that if \( v \in V \) and \( v \neq 0 \), then there exists a quasicharacter \( \vartheta \) of \( N \) such that \( v \notin V(\vartheta) \).

5. Let \( (\pi, V) \) be a smooth representation of \( H \). Suppose that \( V_N = \{0\} \) and \( V_\vartheta = \{0\} \) for some nontrivial quasicharacter \( \vartheta \). Prove that \( V = \{0\} \). (Hint: Consider the action of \( S \) on the spaces \( V(\vartheta) \) for \( \vartheta \) nontrivial.)

6. Let \( \vartheta \) be a nontrivial quasicharacter of \( N \). Let \( \pi = \text{Ind}^H_N \vartheta \) and let \( V \) be the space of \( \pi \). Let \( \pi^\varphi = \text{c-Ind}^H_N \vartheta \) and let \( V^\varphi \) be the space of \( \pi^\varphi \). Show that
   a) \( V(N) = V^\varphi(N) = V^\varphi \) and \( V/V^\varphi(N) = \{0\} \).
   b) The map \( f \mapsto f(1) \) induces isomorphisms \( V_\vartheta \simeq \mathbb{C} \) and \( V^\varphi_\vartheta \simeq \mathbb{C} \).

7. Let \( \vartheta \) be a nontrivial quasicharacter of \( N \).
   a) Prove that \( \text{c-Ind}^H_N \vartheta \) is an irreducible representation of \( H \).
   b) Prove that the contragredient (smooth dual) of \( \text{c-Ind}^H_N \vartheta \) is reducible.
   c) Prove that \( \text{c-Ind}^H_N \vartheta \) is not admissible.

8. Let \( (\pi, V) \) be a smooth representation of \( H \) and let \( \vartheta \) be a nontrivial quasicharacter of \( N \). Let \( q_\vartheta : V \to V_\vartheta \) be the quotient map. Frobenius reciprocity gives an isomorphism \( A : \text{Hom}_H(V, \text{Ind}^H_N V_\vartheta) \simeq \text{Hom}_N(V, V_\vartheta) \). Let \( q_\varphi = A^{-1}(q_\vartheta) \). (That is, for \( v \in V \), \( q_\varphi(v) \) is the function \( h \mapsto q(\pi(h)v) \).) Prove that the \( H \)-morphism \( q_\varphi : V \to \text{Ind}^H_N V_\vartheta \) induces an isomorphism \( V(N) \simeq \text{c-Ind}^H_N V_\vartheta \).

9. Let \( (\pi, V) \) be an irreducible smooth representation of \( H \). Prove that exactly one of the following holds:
   (i) \( \dim V = 1 \) and there exists a quasicharacter \( \chi \) of \( H \) such that \( \pi(hn) = \chi(h) \) for all \( n \in N \) and \( h \in H \).
   (ii) \( V \) is infinite-dimensional and \( \pi \simeq \text{c-Ind}^H_N \vartheta \) for any nontrivial quasicharacter \( \vartheta \) of \( N \).

Describe the spaces \( V_N \) and \( V_\vartheta \) (for \( \vartheta \) nontrivial) in both cases.