1. Let \( u(x, y, t) \) be a solution of the heat equation \( u_t = \alpha^2 (u_{xx} + u_{yy}) \). Let \( U \) be the function defined by

\[
U(X(x, y), Y(x, y), t) = u(x, y, t)
\]

(a) If \( X(x, y) = x + 3 \) and \( Y(x, y) = y - 1 \), what PDE does \( U(X, Y, t) \) satisfy? Give a physical explanation for what you found.

(b) If \( X(x, y) = -x \) and \( Y(x, y) = y \), what PDE does \( U(X, Y, t) \) satisfy? Give a physical explanation for what you found. If the initial data is even in \( x \) about \( x = 0 \) what can you say about the solution \( u(x, y, t) \)? If the initial data is odd in \( x \) about \( x = 3 \) what can you say about the solution \( u(x, y, t) \)? What if the initial data is even in \( y \) about \( y = 0 \)?

(c) If

\[
X(x, y) = \frac{\sqrt{3}}{2} x - \frac{1}{2} y, \quad \text{and} \quad Y(x, y) = \frac{1}{2} x + \frac{\sqrt{3}}{2} x,
\]

what PDE does \( U(X, Y, t) \) satisfy? Give a physical explanation for what you found. If the initial data is rotationally symmetric about \( (x, y) = (0, 0) \) then what can you say about the solution \( u(x, y, t) \)?

(d) If

\[
X(x, y) = 2x, \quad Y(x, y) = 2y, \quad \text{and} \quad T(t) = 4t
\]

what PDE does \( U(X, Y, T) \) satisfy? Give a physical explanation for what you found.

2. In solving the heat equation via separation of variables, Farlow is a bit shallow at the following point. He’s found that

\[
\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = k.
\]

This implies that \( T(t) \) and \( X(x) \) satisfy the ODEs

\[
T'(t) = k\alpha^2 T(t), \quad X''(x) = kX(x).
\]

If the constant \( k \) is a negative number then one can represent \( k \) as \(-\lambda^2\) and the ODEs become

\[
T'(t) = -\lambda^2 \alpha^2 T(t), \quad X''(x) = -\lambda^2 X(x),
\]

resulting in the solution \( X(x) = A \cos(\lambda x) + B \sin(\lambda x) \) for constants \( A \) and \( B \) and a \( T(t) \) that decays to zero as \( t \to \infty \).

If the constant \( k \) is zero then the ODEs become

\[
T'(t) = 0, \quad X''(x) = 0,
\]

resulting in the solution \( X(x) = Ax + B \) for constants \( A \) and \( B \) and a \( T(t) \) that is constant.

If the constant \( k \) is a positive number then one can represent \( k \) as \( \lambda^2 \) and the ODEs become

\[
T'(t) = \lambda^2 \alpha^2 T(t), \quad X''(x) = \lambda^2 X(x),
\]

resulting in the solution \( X(x) = \cos(\lambda x) + B \sin(\lambda x) \).
resulting in the solution $X(x) = A \cosh(\lambda x) + B \sinh(\lambda x)$ for constants $A$ and $B$ and a $T(t)$ that goes to infinity as $t \to \infty$.

The recommended text, DuChateau & Zachmann, has a good discussion of these Sturm-Liouville problems.

The point of the following problem is for you to make sure that Farlow didn’t miss anything in the cases of “clamped” boundary conditions, no-flux boundary conditions, or the boundary conditions that correspond to the rod being in contact with another medium that’s held at a fixed temperature.

(a) If the IBVP has the “clamped” boundary conditions

$$u(0, t) = 0, \quad u(L, t) = 0$$

prove that there are no nontrivial solutions $X(x)$ when $k \geq 0$.

(b) If the IBVP has the no-flux boundary conditions

$$u_x(0, t) = 0, \quad u_x(L, t) = 0$$

prove that there are no nontrivial solutions $X(x)$ when $k > 0$.

(c) Consider the boundary conditions that would correspond to the bar being held in a material, at temperature $T_1$, at one end and in a different material, at temperature $T_2$, at the other end:

$$\begin{align*}
  u_t &= \alpha^2 u_{xx} \\
  -u_x(0, t) &= -\frac{h_1}{k} (u(0, t) - T_1) \\
  u_x(L, t) &= -\frac{h_2}{k} (u(L, t) - T_2) \\
  u(x, 0) &= \phi(x)
\end{align*}$$

i. Why do I have two different $h$s in the boundary conditions?

ii. Find $u_{ss}(x)$, the steady state for the IBVP. Is there always a steady state? Write $u_{ss}(0)$ as $a_{11} T_1 + a_{12} T_2$ and $u_{ss}(L)$ as $a_{21} T_1 + a_{22} T_2$. Is there a relationship between $a_{11}$ and $a_{12}$? Between $a_{21}$ and $a_{22}$? Is there a relationship between the pair $(a_{11}, a_{12})$ and the pair $(a_{21}, a_{22})$? If $h_1 = 0$, what is $u_{ss}$? Does your answer make sense? If $h_2 = 0$, what is $u_{ss}$? Does your answer make sense?

iii. Subtract the steady state from $u(x, t)$ to find an IBVP for $U(x, t)$:

$$\begin{align*}
  U_t &= \alpha^2 U_{xx} \\
  -U_x(0, t) &= -\frac{h_1}{k} U(0, t) \\
  U_x(L, t) &= -\frac{h_2}{k} U(L, t) \\
  u(x, 0) &= \phi(x) - u_{ss}(x)
\end{align*}$$

We seek to solve this IBVP by separation of variables. Prove that there are no nontrivial solutions $X(x)$ when $k > 0$. What about if $k = 0$?
(d) In the previous problem set, you considered the IBVP

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < 2, \ 0 < t \\
\frac{\partial u}{\partial x}(0, t) + 3u(0, t) &= 0 \quad \text{for } 0 < t \\
\frac{\partial u}{\partial x}(2, t) - 8u(2, t) &= 0 \quad \text{for } 0 < t \\
u(x, 0) &= \phi(x)
\end{align*}
\]

There are two nontrivial \(X(x)\) which correspond to \(k > 0\) solutions. Find them.