Mat 457/1000, Term Test, January 8, 2003

There are five warm–up problems, each worth 16 points. Then, there are four harder problems, each worth 26 points. Please read all nine problems and choose five to write up.

<table>
<thead>
<tr>
<th>five warm–up problems</th>
<th>80 points possible</th>
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<tbody>
<tr>
<td>four warm–problems</td>
<td>90 points possible</td>
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<tr>
<td>three warm–problems</td>
<td>100 points possible</td>
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<tr>
<td>two warm–problems</td>
<td>110 points possible</td>
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<tr>
<td>one warm–problem</td>
<td>120 points possible</td>
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I will only grade five answers! I want you to have plenty of things to choose from, but I also want you to have plenty of time to work the problems and write them up carefully. **Please do not write up more than five problems!!**

Warm-up problems

WU1: Let $L$ be a real vector space and let $M \subset L$ be a convex body whose interior contains 0. We define the Minkowski functional, $p_M : L \to [0, \infty]$ by

$$p_M(x) = \inf \{ r > 0 \mid x/r \in M \}$$

a) Let $L = \mathbb{R}^2$ and let $M$ be the square with vertices $(\pm2, \pm2)$. Find an (explicit) formula for $p_M$. Plot the level set $\{x \mid p_M(x) = 2\}$.

b) Let $L = \mathbb{R}^2$ and let $M$ be the infinite strip bounded above by $y = 1$ and below by $y = -1$. Find an (explicit) formula for $p_M$. Plot the level set $\{x \mid p_M(x) = 2\}$.

c) Prove $p_M(x) < \infty, \forall x \in L, p_M(x + y) \leq p_M(x) + p_M(y), \forall x, y \in L$, and $p_M(\alpha x) = \alpha p_M(x), \forall \alpha > 0$.

WU2: Let $(L, \| \cdot \|)$ be a normed real vector space. Fix $x_0 \in L$. Find $f \in L^*$ such that $f(10 \cdot x_0) = 5$ and such that

$$\|f\|_{L^*} = \frac{1}{2\|x_0\|_L}$$

WU3: Let $(L, \langle \cdot, \cdot \rangle)$ be a Euclidean space and let $\{\phi_k\}_{k=1}^n$ be an orthogonal family in $L$. Fix $f \in L$. Find $a_1, a_2, \ldots, a_n$ so that

$$\sum_{k=1}^n a_k \phi_k$$

is the closest member of $\text{span}\{\phi_1, \ldots, \phi_n\}$ to $f$.

WU4: Give an example of a separable topological vector space $(L, \tau)$ for which the dual space $L^*$ endowed with the strong topology $b$ is not separable. (Recall, a topological space is separable if it contains a countable dense subset.)
WU5: Let \((X, \tau)\) be a topological vector space. Let \(X^*\) be the (vector) space of continuous (with respect to \(\tau\)) real-valued linear functionals on \(X\).

a) Define the weak* topology on \(X^*\).

b) Prove the following: A sequence \(\{f_n\}\) of elements in \(X^*\) is weak* convergent to \(f_0 \in X^*\) if and only if

\[
\lim_{n \to \infty} f_n(x) = f_0(x) \quad \forall x \in X.
\]

**Harder problems**

H1: Let \((L, \| \cdot \|)\) be an infinite-dimensional Banach space. Consider the closed ball

\[
B = \{x \mid \|x\| \leq 1\} \subset L.
\]

Prove \(B\) is not compact.

H2: Let \(L = l^2(\mathbb{C}, \mathbb{N})\) with

\[
\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k, \quad \text{and} \quad \|x\|_L = \sqrt{\sum_{k=1}^{\infty} |x_k|^2}.
\]

Let \(L^*\) be the dual of \(L\), endowed with the strong topology. Prove that

\[(L^*, \| \cdot \|_{L^*}) \leftrightarrow (L, \| \|_L).
\]

That is, find a one-to-one and onto mapping \(\pi : L \to L^*\) such that

\[\|\pi(x)\|_{L^*} = \|x\|_L\]

H3: A topological vector space \((L, \tau)\) is **locally convex** if every nonempty open set contains a nonempty convex open set. Let \((L, \tau)\) be a locally convex topological vector space and let \(x \in L\). Prove that if \(U\) is an open set containing \(x\), then there is a convex open set \(V\) such that \(x \in V \subset U\).

H4: Let \((L, \tau)\) be a real topological vector space that satisfies two additional properties:

1) it is locally convex and

2) it satisfies the first axiom of separability. (It is \(T_1\): given \(x \neq y\) there is an open neighborhood of \(x\) that doesn’t contain \(y\) and there is an open neighborhood of \(y\) that doesn’t contain \(x\).

Prove that \(L^*\) has **sufficiently many** elements. That is, given \(x \neq y\) there is \(f \in L^*\) with \(f(x) \neq f(y)\).