Warm-up exercises

The two of you who’re writing up warm-up exercises should write up the following problems: Kolmogorov & Fomin: 2, 4, 6, 7, 12, mine: 1, 5, 6, 7, 10, 11, 14.

1. Let \( X = \{-1, 0, 1\} \) and let \( T = \{X, \emptyset, \{1\}, \{-1\}, \{-1, 1\}\}. \) Show that the axioms for a topological space are satisfied. List all continuous functions \( f : X \to X. \)

2. On every set \( X \) there are two trivial topologies: \( T_1 = \{\emptyset, X\} \) and \( T_2 = \) the collection of all subsets of \( X. \) For each of these topologies, find the smallest open set containing a given point \( x \in X. \)

3. Let \( (X, T) \) be a topological space, and let \( A \subseteq X. \) Supposed that for each \( x \in A \) there is \( U_x \in T \) such that \( x \in U_x \subseteq A. \) Show that \( A \in T. \)

4. Let \( (X, T) \) be a set with two metrics \( \rho_1 \) and \( \rho_2. \) Let \( T_1 \) and \( T_2 \) be the topologies induced by \( \rho_1 \) and \( \rho_2 \) respectively. Show that \( T_1 = T_2 \) if and only if \( \rho_1 \) and \( \rho_2 \) are equivalent metrics.

5. Let \( X = \mathbb{R}^2 \) and let \( G \) be the set of all ellipses in the plane whose major axis is parallel to the \( x \)-axis and whose major axis is twice as long as the minor axis. Prove that \( G \) is a basis for the \( l^2 \) metric topology on \( X. \) Prove that it is a basis for the \( l^p \) metric topology on \( X. \)

6. Consider the following three definitions of continuity. The first definition is a “topological” definition and the second definition is an “\( \epsilon - \delta \)” definition. The third is the “sequential” definition.

**Definition 1.** Let \( (X, T) \) and \( (Y, T') \) be topological spaces and let \( f : X \to Y \) be a function. One says that \( f \) is continuous at \( x_0 \in X \) if for every open set \( V \subseteq Y \) that contains \( f(x_0) \) there is an open set \( U \subseteq X \) containing \( x_0 \) so that \( f(U) \subseteq V. \) Finally, \( f \) is continuous if it is continuous at each \( x \in X. \)

**Definition 2.** Let \( N_x \) and \( M_y \) be local bases at \( x \) and \( y \) for the sets \( X \) and \( Y \) respectively. We say that the function \( f : X \to Y \) is continuous at \( x_0 \) if for every \( V \in M_{f(x_0)} \) there is a \( U \in N_{x_0} \) such that \( f(U) \subseteq V. \) Finally, \( f \) is continuous if it is continuous at each \( x \in X. \)

**Definition 3.** Let \( (X, T) \) and \( (Y, T') \) be topological spaces and let \( f : X \to Y \) be a function. One says that \( f \) is continuous at \( x_0 \in X \) if for every sequence \( \{x_n\} \) that converges to \( x_0, \) the image sequence \( \{f(x_n)\} \) converges to \( f(x_0). \) Finally, \( f \) is continuous if it is continuous at each \( x \in X. \)

Prove that if \( T \) is the topology on \( X \) generated by the local bases \( N_x \) and \( T' \) is the topology on \( Y \) generated by the local bases \( M_y, \) then \( f : X \to Y \) is continuous by according to definition 1 if and only if it is continuous according to definition 2.

Prove that if \( f \) is continuous according to definition 1 then it continuous according to definition 3. Prove that if \( (X, T) \) is first countable and \( f \) is continuous according to definition 3 then it is continuous according to definition 1.

7. Show that the constant functions are the only continuous functions from \( \mathbb{R} \) with the Zariski topology to \( \mathbb{R} \) with the usual metric topology (based on \( \rho(x, y) = |x - y|). \)
8. Prove the following about the function \( f : X \to Y \):
   a) \( f^{-1}(\cup_a A_a) = \cup_a f^{-1}(A_a) \).
   b) \( f^{-1}(\cap_a A_a) = \cap_a f^{-1}(A_a) \).
   c) \( f(f^{-1}(A)) \subseteq A \)
   And show it can be a proper subset.
   d) \( B \subseteq f^{-1}(f(B)) \)
   And show it can be a proper subset.

9. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function whose inverse is also a function. Prove or disprove:
   \( f^{-1} \) is continuous.

10. Prove that if \( (X, T) \) is Hausdorff, then for every \( x \) the set \( \{x\} \) is closed. Is the converse true?

11. Prove that in a Hausdorff space, every convergent sequence has a unique limit.

12. Let \( (X, T) \) be first countable and assume that every convergent sequence has a unique limit.
    Show that \( (X, T) \) is also Hausdorff.

13. Assume \( (X, T) \) and \( (Y, T') \) are homeomorphic and \( X \) is Hausdorff. Prove that \( Y \) is Hausdorff.

14. Let \( M \) be the set of real \( n \times n \) matrices, equipped with the metric

   \[
   \rho(A, B) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} (A_{ij} - B_{ij})^2}
   \]

   a) Prove that the determinant function \( \det : M \to \mathbb{R} \) is continuous.
   b) Show that the set of invertible matrices is open but not connected.

15. Problems 1-12 on pages 90-91 of Kolmogorov and Fomin.

**Homework Problems**

1. Problem 13 on page 91 of Kolmogorov and Fomin.

2. Problem 14 on page 92 of Kolmogorov and Fomin.

3. Let \( X \) be the set of all infinitely differentiable functions on \( \mathbb{R} \). Let \( \phi \in X \). To characterise when \( \phi \in X \) is close to \( \phi \), we should — since the existence of all derivatives is a characteristic of the space \( X \) — introduce a measure of the difference between the derivatives of \( \psi \) and those of \( \phi \). We do that as follows. The analogue of the spheres \( S(x, r) \) will be the sets

   \[
   U(\phi, m, R, \epsilon) = \{ \psi \in X | \max_{0 \leq k \leq m} \sup_{-R \leq x \leq R} |\phi^{(k)}(x) - \psi^{(k)}(x)| < \epsilon \}
   \]

   where \( \phi^{(k)} \) denotes the \( k \)th derivative of \( \phi \). In other words, \( \psi \) belongs to \( U(\phi, m, R, \epsilon) \) if and only if its first \( m \) derivatives are \( \epsilon \)-close to those of \( \phi \) in the \( l^\infty \) metric on \([-R, R] \).
A local base is then defined by

\[ N(\phi) = \{ U(\phi, m, R, \epsilon) \mid m \in \mathbb{N}, R > 0, \epsilon > 0 \}. \]

a) Verify that \( U(\phi, m', R', \epsilon') \subset U(\phi, m, R, \epsilon) \) if \( m' \geq m, \ R' \geq R \) and \( \epsilon' \leq \epsilon. \)
b) Verify that \( N(\phi) \) satisfies the conditions of being a local base.
c) Let \( T \) be the topology induced by this local base. Prove that \((X, T)\) is first countable. Is it second countable?
d) One’s knee-jerk reaction is probably to introduce the obvious metric on \( X \):

\[ \rho(\phi, \psi) = \sup_{k \geq 0} \sup_{x \in \mathbb{R}} |\phi^{(k)}(x) - \psi^{(k)}(x)| \]

Why doesn’t this work?
e) Consider the function \( f : X \to \mathbb{R} \) defined by

\[ f(\phi) = \int_{-5}^{8} \left( \phi^{(3)}(x) \right)^3 \, dx. \]

Prove that \( f \) is a continuous function.

4. The system of ordinary differential equations

\[ \begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= x
\end{align*} \]

has general solution \( x(t) = a \cosh t + b \sinh t, \ y(t) = a \sinh t + b \cosh t \). The orbits are the curves \( \{(x(t), y(t)) \mid t \in \mathbb{R}\} \) in \( \mathbb{R}^2 \). They are various branches and limiting cases of the hyperbolas \( x^2 - y^2 = a^2 - b^2 \). (Draw them!)

Let \( X \) be the set of orbits. We give \( X \) a topology as follows: a set \( U \subset X \) is open if the set of all initial conditions in \( \mathbb{R}^2 \) that give rise to the orbits in \( U \) is open in \( \mathbb{R}^2 \) with the \( l^2 \) metric topology. Prove that this is a topology.

Prove that \((X, T)\) is not Hausdorff.

5. Let \( M \) be the set of real invertible matrices. Let \( m \in M \). The orbit through \( m \) is defined to be the set of all matrices \( g \) that are similar to \( m \) (written \( g \sim m \)). I.e., the set \( \{ p \in M \mid p = g mg^{-1} \text{ for some } g \in M \} \). Let \( X \) be the set of orbits.

a) List all the elements of \( X \) for the case of \( 2 \times 2 \) invertible matrices.
b) A subset \( U \subset X \) is defined to be open if the set \( \{ m \in M \mid m \text{ lies on one of the orbits in } U \} \) is open in \( M \) with the metric topology given in warm-up problem 14. Is \((X, T)\) a Hausdorff space?