THE STONE-WEIERSTRASS THEOREM

7.26 Theorem If $f$ is a continuous complex function on $[a, b]$, there exists a sequence of polynomials $P_n$ such that

$$\lim_{n \to \infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If $f$ is real, the $P_n$ may be taken real.

This is the form in which the theorem was originally discovered by Weierstrass.

Proof We may assume, without loss of generality, that $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$. For if the theorem is proved for this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \quad (0 \leq x \leq 1).$$

Here $g(0) = g(1) = 0$, and if $g$ can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for $f$, since $f - g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for $x$ outside $[0, 1]$. Then $f$ is uniformly continuous on the whole line.

We put

$$Q_n(x) = c_n(1 - x^2)^n \quad (n = 1, 2, 3, \ldots),$$

where $c_n$ is chosen so that

$$\int_{-1}^{1} Q_n(x) \, dx = 1 \quad (n = 1, 2, 3, \ldots).$$

We need some information about the order of magnitude of $c_n$. Since

$$\int_{-1}^{1} (1 - x^2)^n \, dx = 2 \int_{0}^{1} (1 - x^2)^n \, dx \geq 2 \int_{0}^{1/\sqrt{n}} (1 - x^2)^n \, dx$$

$$\geq 2 \int_{0}^{1/\sqrt{n}} (1 - nx^2) \, dx$$

$$= \frac{4}{3\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}},$$

it follows from (48) that

$$c_n < \sqrt{n}. \quad (49)$$
The inequality \((1 - x^2)^n \geq 1 - nx^2\) which we used above is easily shown to be true by considering the function
\[
(1 - x^2)^n - 1 + nx^2
\]
which is zero at \(x = 0\) and whose derivative is positive in \((0, 1)\).

For any \(\delta > 0\), (49) implies
\[
Q_n(x) \leq \sqrt{n} (1 - \delta^2)^n \quad (\delta \leq |x| \leq 1),
\]
so that \(Q_n \to 0\) uniformly in \(\delta \leq |x| \leq 1\).

Now set
\[
P_n(x) = \int_{-1}^{1} f(x + t)Q_n(t) \, dt \quad (0 \leq x \leq 1).
\]

Our assumptions about \(f\) show, by a simple change of variable, that
\[
P_n(x) = \int_{-x}^{1-x} f(x + t)Q_n(t) \, dt = \int_{0}^{1} f(t)Q_n(t - x) \, dt,
\]
and the last integral is clearly a polynomial in \(x\). Thus \(\{P_n\}\) is a sequence of polynomials, which are real if \(f\) is real.

Given \(\varepsilon > 0\), we choose \(\delta > 0\) such that \(|y - x| < \delta\) implies
\[
|f(y) - f(x)| < \frac{\varepsilon}{2}.
\]

Let \(M = \sup |f(x)|\). Using (48), (50), and the fact that \(Q_n(x) \geq 0\), we see that for \(0 \leq x \leq 1\),
\[
|P_n(x) - f(x)| = \left| \int_{-1}^{1} [f(x + t) - f(x)]Q_n(t) \, dt \right|
\leq \int_{-1}^{1} |f(x + t) - f(x)|Q_n(t) \, dt
\leq 2M \int_{-\delta}^{\delta} Q_n(t) \, dt + \frac{\varepsilon}{2} \int_{\delta}^{\delta} Q_n(t) \, dt + 2M \int_{\delta}^{1} Q_n(t) \, dt
\leq 4M \sqrt{n} (1 - \delta^2)^n + \frac{\varepsilon}{2}
\leq \varepsilon
\]
for all large enough \(n\), which proves the theorem.

It is instructive to sketch the graphs of \(Q_n\) for a few values of \(n\); also, note that we needed uniform continuity of \(f\) to deduce uniform convergence of \(\{P_n\}\).
In the proof of Theorem 7.32 we shall not need the full strength of Theorem 7.26, but only the following special case, which we state as a corollary.

**7.27 Corollary** For every interval \([-a, a]\) there is a sequence of real polynomials \(P_n\) such that \(P_n(0) = 0\) and such that

\[
\lim_{n \to \infty} P_n(x) = |x|
\]

uniformly on \([-a, a]\).

**Proof** By Theorem 7.26, there exists a sequence \(\{P_n^*\}\) of real polynomials which converges to \(|x|\) uniformly on \([-a, a]\). In particular, \(P_n^*(0) \to 0\) as \(n \to \infty\). The polynomials

\[
P_n(x) = P_n^*(x) - P_n^*(0) \quad (n = 1, 2, 3, \ldots)
\]

have desired properties.

We shall now isolate those properties of the polynomials which make the Weierstrass theorem possible.

**7.28 Definition** A family \(\mathcal{A}\) of complex functions defined on a set \(E\) is said to be an algebra if (i) \(f + g \in \mathcal{A}\), (ii) \(fg \in \mathcal{A}\), and (iii) \(cf \in \mathcal{A}\) for all \(f, g \in \mathcal{A}\) and for all complex constants \(c\), that is, if \(\mathcal{A}\) is closed under addition, multiplication, and scalar multiplication. We shall also have to consider algebras of real functions; in this case, (iii) is of course only required to hold for all real \(c\).

If \(\mathcal{A}\) has the property that \(f \in \mathcal{A}\) whenever \(f_n \in \mathcal{A}\) \((n = 1, 2, 3, \ldots)\) and \(f_n \to f\) uniformly on \(E\), then \(\mathcal{A}\) is said to be uniformly closed.

Let \(\mathcal{B}\) be the set of all functions which are limits of uniformly convergent sequences of members of \(\mathcal{A}\). Then \(\mathcal{B}\) is called the uniform closure of \(\mathcal{A}\). (See Definition 7.14.)

For example, the set of all polynomials is an algebra, and the Weierstrass theorem may be stated by saying that the set of continuous functions on \([a, b]\) is the uniform closure of the set of polynomials on \([a, b]\).

**7.29 Theorem** Let \(\mathcal{B}\) be the uniform closure of an algebra \(\mathcal{A}\) of bounded functions. Then \(\mathcal{B}\) is a uniformly closed algebra.

**Proof** If \(f \in \mathcal{B}\) and \(g \in \mathcal{B}\), there exist uniformly convergent sequences \(\{f_n\}, \{g_n\}\) such that \(f_n \to f, g_n \to g\) and \(f_n \in \mathcal{A}, g_n \in \mathcal{A}\). Since we are dealing with bounded functions, it is easy to show that

\[
f_n + g_n \to f + g, \quad f_n g_n \to fg, \quad cf_n \to cf,
\]

where \(c\) is any constant, the convergence being uniform in each case.

Hence \(f + g \in \mathcal{B}, fg \in \mathcal{B}, \) and \(cf \in \mathcal{B}\), so that \(\mathcal{B}\) is an algebra.

By Theorem 2.27, \(\mathcal{B}\) is (uniformly) closed.
7.30 Definition Let $\mathcal{A}$ be a family of functions on a set $E$. Then $\mathcal{A}$ is said to separate points on $E$ if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in \mathcal{A}$ such that $g(x) \neq 0$, we say that $\mathcal{A}$ vanishes at no point of $E$.

The algebra of all polynomials in one variable clearly has these properties on $R^1$. An example of an algebra which does not separate points is the set of all even polynomials, say on $[-1, 1]$, since $f(-x) = f(x)$ for every even function $f$.

The following theorem will illustrate these concepts further.

7.31 Theorem Suppose $\mathcal{A}$ is an algebra of functions on a set $E$, $\mathcal{A}$ separates points on $E$, and $\mathcal{A}$ vanishes at no point of $E$. Suppose $x_1, x_2$ are distinct points of $E$, and $c_1, c_2$ are constants (real if $\mathcal{A}$ is a real algebra). Then $\mathcal{A}$ contains a function $f$ such that

$$f(x_1) = c_1, \quad f(x_2) = c_2.$$  

Proof The assumptions show that $\mathcal{A}$ contains functions $g, h, k$ such that

$$g(x_1) \neq g(x_2), \quad h(x_1) \neq 0, \quad k(x_2) \neq 0.$$  

Put

$$u = gk - g(x_1)k, \quad v = gh - g(x_2)h.$$  

Then $u \in \mathcal{A}, v \in \mathcal{A}, u(x_1) = v(x_2) = 0, u(x_2) \neq 0$, and $v(x_1) \neq 0$. Therefore

$$f = \frac{c_1v}{v(x_1)} + \frac{c_2u}{u(x_2)}$$  

has the desired properties.

We now have all the material needed for Stone's generalization of the Weierstrass theorem.

7.32 Theorem Let $\mathcal{A}$ be an algebra of real continuous functions on a compact set $K$. If $\mathcal{A}$ separates points on $K$ and if $\mathcal{A}$ vanishes at no point of $K$, then the uniform closure $\mathcal{B}$ of $\mathcal{A}$ consists of all real continuous functions on $K$.

We shall divide the proof into four steps.

STEP 1 If $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$.

Proof Let

$$a = \sup |f(x)| \quad (x \in K)$$  

(52)
and let $\varepsilon > 0$ be given. By Corollary 7.27 there exist real numbers $c_1, \ldots, c_n$ such that

$$|\sum_{i=1}^{n} c_i y^i - |y| < \varepsilon \quad (-a \leq y \leq a).$$

Since $\mathcal{B}$ is an algebra, the function

$$g = \sum_{i=1}^{n} c_i f^i$$

is a member of $\mathcal{B}$. By (52) and (53), we have

$$|g(x) - |f(x)|| < \varepsilon \quad (x \in K).$$

Since $\mathcal{B}$ is uniformly closed, this shows that $|f| \in \mathcal{B}$.

STEP 2 If $f \in \mathcal{B}$ and $g \in \mathcal{B}$, then $\max (f, g) \in \mathcal{B}$ and $\min (f, g) \in \mathcal{B}$.

By $\max (f, g)$ we mean the function $h$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \geq g(x), \\ g(x) & \text{if } f(x) < g(x), \end{cases}$$

and $\min (f, g)$ is defined likewise.

**Proof** Step 2 follows from step 1 and the identities

$$\max (f, g) = \frac{f + g}{2} + \frac{|f - g|}{2},$$

$$\min (f, g) = \frac{f + g}{2} - \frac{|f - g|}{2}.$$

By iteration, the result can of course be extended to any finite set of functions: If $f_1, \ldots, f_n \in \mathcal{B}$, then $\max (f_1, \ldots, f_n) \in \mathcal{B}$, and

$$\min (f_1, \ldots, f_n) \in \mathcal{B}.$$

STEP 3 Given a real function $f$, continuous on $K$, a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathcal{B}$ such that $g_x(x) = f(x)$ and

$$g_x(t) > f(t) - \varepsilon \quad (t \in K).$$

**Proof** Since $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A}$ satisfies the hypotheses of Theorem 7.31 so does $\mathcal{B}$. Hence, for every $y \in K$, we can find a function $h_y \in \mathcal{B}$ such that

$$h_y(x) = f(x), \quad h_y(y) = f(y).$$
By the continuity of $h_y$ there exists an open set $J_y$, containing $y$, such that

$$h_y(t) > f(t) - \varepsilon \quad (t \in J_y).$$

(56)

Since $K$ is compact, there is a finite set of points $y_1, \ldots, y_n$ such that

$$K \subseteq J_{y_1} \cup \cdots \cup J_{y_n}.$$  

(57)

Put

$$g_x = \max (h_{y_1}, \ldots, h_{y_n}).$$

By step 2, $g \in \mathcal{B}$, and the relations (55) to (57) show that $g_x$ has the other required properties.

**STEP 4**  Given a real function $f$, continuous on $K$, and $\varepsilon > 0$, there exists a function $h \in \mathcal{B}$ such that

$$|h(x) - f(x)| < \varepsilon \quad (x \in K).$$

(58)

Since $\mathcal{B}$ is uniformly closed, this statement is equivalent to the conclusion of the theorem.

**Proof**  Let us consider the functions $g_x$, for each $x \in K$, constructed in step 3. By the continuity of $g_x$, there exist open sets $V_x$ containing $x$, such that

$$g_x(t) < f(t) + \varepsilon \quad (t \in V_x).$$

(59)

Since $K$ is compact, there exists a finite set of points $x_1, \ldots, x_m$ such that

$$K \subseteq V_{x_1} \cup \cdots \cup V_{x_m}.$$  

(60)

Put

$$h = \min (g_{x_1}, \ldots, g_{x_m}).$$

By step 2, $h \in \mathcal{B}$, and (54) implies

$$h(t) > f(t) - \varepsilon \quad (t \in K),$$

(61)

whereas (59) and (60) imply

$$h(t) < f(t) + \varepsilon \quad (t \in K).$$

(62)

Finally, (58) follows from (61) and (62).
Theorem 7.32 does not hold for complex algebras. A counterexample is

given in Exercise 21. However, the conclusion of the theorem does hold, even

for complex algebras, if an extra condition is imposed on \( \mathcal{A} \), namely, that \( \mathcal{A} \)

be self-adjoint. This means that for every \( f \in \mathcal{A} \) its complex conjugate \( \overline{f} \)

must also belong to \( \mathcal{A} \); \( \overline{f} \) is defined by \( \overline{f}(x) = \overline{f(x)} \).

7.33 Theorem Suppose \( \mathcal{A} \) is a self-adjoint algebra of complex continuous

functions on a compact set \( K \), \( \mathcal{A} \) separates points on \( K \), and \( \mathcal{A} \) vanishes at no

point of \( K \). Then the uniform closure \( \mathcal{B} \) of \( \mathcal{A} \) consists of all complex continuous

functions on \( K \). In other words, \( \mathcal{A} \) is dense \( \mathcal{C}(K) \).

Proof Let \( \mathcal{A}_R \) be the set of all real functions on \( K \) which belong to \( \mathcal{A} \).

If \( f \in \mathcal{A} \) and \( f = u + iv \), with \( u, v \) real, then \( 2u = f + \overline{f} \), and since \( \mathcal{A} \)

is self-adjoint, we see that \( u \in \mathcal{A}_R \). If \( x_1 \neq x_2 \), there exists \( f \in \mathcal{A} \)

such that \( f(x_1) = 1, f(x_2) = 0 \); hence \( 0 = u(x_2) = u(x_1) = 1 \), which shows that

\( \mathcal{A}_R \) separates points on \( K \). If \( x \in K \), then \( g(x) \neq 0 \) for some \( g \in \mathcal{A} \), and

there is a complex number \( \lambda \) such that \( \lambda g(x) > 0 \); if \( f = \lambda g, f = u + iv \), it

follows that \( u(x) > 0 \); hence \( \mathcal{A}_R \) vanishes at no point of \( K \).

Thus \( \mathcal{A}_R \) satisfies the hypotheses of Theorem 7.32. It follows that

every real continuous function on \( K \) lies in the uniform closure of \( \mathcal{A}_R \),

hence lies in \( \mathcal{B} \). If \( f \) is a complex continuous function on \( K \), \( f = u + iv \),

then \( u \in \mathcal{B}, v \in \mathcal{B} \), hence \( f \in \mathcal{B} \). This completes the proof.

**EXERCISES**

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

2. If \( \{f_n\} \) and \( \{g_n\} \) converge uniformly on a set \( E \), prove that \( \{f_n + g_n\} \) converges uniformly on \( E \). If, in addition, \( \{f_n\} \) and \( \{g_n\} \) are sequences of bounded functions, prove that \( \{f_n g_n\} \) converges uniformly on \( E \).

3. Construct sequences \( \{f_n\}, \{g_n\} \) which converge uniformly on some set \( E \), but such that \( \{f_n g_n\} \) does not converge uniformly on \( E \) (of course, \( \{f_n g_n\} \) must converge on \( E \)).

4. Consider

\[
 f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x^2} .
\]

For what values of \( x \) does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is \( f \)
continuous wherever the series converges? Is \( f \) bounded?