You may not use calculators, cell phones, or PDAs during the exam. Partial credit will be given for partially correct work. Please read through the entire test before starting, and take note of how many points each question is worth. Please put a box around your solutions so that the grader may find them easily.

Family name:

Given name(s):

Answer key

Please sign here:

Student ID number:

| Problem 1: | /10 |
| Problem 2: | /25 |
| Problem 3: | /15 |
| Problem 4: | /10 |
| Problem 5: | /10 |
| Problem 6: | /10 |
| Problem 7: | /10 |
| Problem 8: | /10 |
| Total:     | /100 |
1. a. (5 points) Use graphical methods to solve the problem:

Maximize $\mathbf{c}^T \mathbf{x}$ subject to

$$A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq 0,$$

where

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

LP:

maximize $2x_1 - 2x_2$

subject to

$-x_1 + x_2 \leq 1$

$x_1 - x_2 \leq 1$

$x_1, x_2 \geq 0$

Since $-x_1 + x_2 \leq 1 \Rightarrow x_2 \leq x_1 + 1$

Since $x_1 - x_2 \leq 1 \Rightarrow x_1 - 1 \leq x_2$

so feasible region is

![Graphical representation of the feasible region](image)

The level sets of the objective function are

$2x_1 - 2x_2 = k \Rightarrow 2x_1 - k = 2x_2 \Rightarrow x_2 = x_1 - \frac{k}{2}.$

These are lines parallel to the two lines that bound the feasible region. So the maximum and minimum are achieved, both at infinitely many points. $k = 2$ at $y = x - 1$, $k = -2$ at $y = x + 1$.

max value = 2 achieved at $y = x - 1, x \geq 0$
b. (5 points) State the dual problem and use graphical methods to solve it.

The dual problem is

\[
\begin{align*}
\text{minimize} & \quad w_1 + w_2 \\
\text{subject to} & \quad -w_1 + w_2 \geq 2 \\
& \quad w_1 - w_2 \geq -2 \\
& \quad w_1, w_2 \geq 0
\end{align*}
\]

The constraint set can be written as

\[
\begin{align*}
-w_1 + w_2 \geq 2 \\
-w_1 + w_2 \leq 2 
\end{align*}
\]

\[
\Rightarrow -w_1 + w_2 = 2
\]

\[W_2\]

\[W_1\]

\[\text{feasible solutions}\]

The minimum of \( w_1 + w_2 \) occurs at \( w_1 = 0, w_2 = 2 \), min. value = 2
2. (25 points) Find the optimal solution \((x_1, x_2, x_3)\) of the following problem:

Minimize \(2x_1 - x_2 + 5x_3\) subject to

\[
\begin{align*}
x_1 + x_3 & \geq 2 \\
2x_1 + x_2 &= 2 \\
8x_1 + 3x_2 + 2x_3 &= 10
\end{align*}
\]

where \(x_1 \geq 0, x_2 \leq 0,\) and \(x_3\) is unrestricted.

This is a 25 point problem. If you're having problems or are running out of time, for 15 points you may solve the problem with the constraints \(x_1, x_2, x_3 \geq 0\) instead.

I want to use the simplex method. This requires that the variables be \(\geq 0\). So, since \(x_2 \leq 0\), I'll replace \(x_2\) by \(-x_2\). Also, I'll replace \(x_3\) by \(x_3 - x_4\).

I have to be careful to translate the optimal solution (find back to the original \(x_1(\geq 0), x_2(\leq 0),\) and \(x_3\) unrestricted)!

Minimize \(2x_1 + x_2 + 5x_3 - 5x_4\)

subject to:

\[
\begin{align*}
x_1 + x_3 - x_4 & \geq 2 \\
2x_1 - x_2 &= 2 \\
8x_1 - 3x_2 + 2x_3 - 2x_4 &= 10
\end{align*}
\]

\(x_1, x_2, x_3, x_4 \geq 0\)

Introduce slack variable \(x_5\) into constraint 1.

Introduce three artificial variables: \(y_1, y_2, y_3\)
Minimize \[ 2x_1 + x_2 + 5x_3 - 5x_4 \]
subject to
\[
\begin{align*}
x_1 + x_3 - x_4 - x_5 + y_1 &= 2 \\
2x_1 - x_2 + y_2 &= 2 \\
8x_1 - 3x_2 + 2x_3 - 2x_4 + y_3 &= 10
\end{align*}
\]

To begin phase 1, I want to maximize a different objective function: \(-y_1 - y_2 - y_3\). I need to write it in terms of the nonbasic variables \(x_1, x_2, x_3, x_4, x_5\). I use the equations to do this.

\[
\begin{align*}
e_1 &\Rightarrow -y_1 = x_1 + x_3 - x_4 - x_5 - 2 \\
e_2 &\Rightarrow -y_2 = 2x_1 - x_2 - 2 \\
e_3 &\Rightarrow -y_3 = 8x_1 - 3x_2 + 2x_3 - 2x_4 - 10
\end{align*}
\]

So the objective function \(f\) I am maximizing is
\[
11x_1 - 4x_2 + 3x_3 - 3x_4 - x_5 - 14
\]

Initial tableau:

\[
\begin{array}{cccccccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & \text{RHS} \\
\hline
y_1 & 1 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 2 \\
y_2 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \\
y_3 & 8 & -3 & 2 & -2 & 0 & 0 & 0 & 1 & 10 \\
\hline
1 & -11 & 4 & -3 & 3 & 1 & 0 & 0 & 0 & -14
\end{array}
\]

\(x_1\) incoming, \(y_2\) departing.
\[
\begin{array}{cccccccc}
X_1 & X_2 & X_3 & X_4 & X_5 & Y_1 & Y_2 & Y_3 \\
\downarrow & & & & & & & \\
Y_1 & Y_2 & 1 & -1 & -1 & 1 & -\frac{1}{2} & 0 & 1 \\
X_1 & 1 & -Y_2 & 0 & 0 & 0 & 0 & Y_2 & 0 & 1 \\
Y_3 & 0 & 1 & 2 & -2 & 0 & 0 & -1 & 1 & 2 \\
\hline \\
0 & -\frac{3}{2} & -3 & 3 & 1 & 0 & \frac{1}{2} & 0 & 1 & -3 \\
\end{array}
\]

\[4 - \frac{n}{2} = \frac{8 - 11}{2} = -\frac{3}{2}\]

\[x_3 \text{ entering, } y_1 \text{ departing}\]

\[
\begin{array}{cccccccc}
X_1 & X_2 & X_3 & X_4 & X_5 & Y_1 & Y_2 & Y_3 \\
\downarrow & & & & & & & \\
X_3 & 0 & Y_2 & 1 & -1 & -1 & 1 & -\frac{1}{2} & 0 & 1 \\
X_1 & 1 & -Y_2 & 0 & 0 & 0 & 0 & Y_2 & 0 & 1 \\
Y_3 & 0 & 0 & 0 & 0 & 2 & -2 & -3 & 1 & 0 \\
\hline \\
0 & 0 & 0 & 0 & -2 & 3 & 4 & 0 & 0 & 0 \\
\end{array}
\]

\[x_5 \text{ entering, } y_3 \text{ departing}\]

\[
\begin{array}{cccccccc}
X_1 & X_2 & X_3 & X_4 & X_5 & Y_1 & Y_2 & Y_3 \\
\downarrow & & & & & & & \\
X_3 & 0 & Y_2 & 1 & -1 & 0 & 0 & -2 & Y_2 & 1 \\
X_1 & 1 & -Y_2 & 0 & 0 & 0 & 0 & Y_2 & 0 & 1 \\
X_5 & 0 & 0 & 0 & 0 & 1 & -1 & -\frac{3}{2} & Y_2 & 0 \\
\hline \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{array}
\]

phase 1 terminates!

\[x_1 = 1, x_3 = 1, x_5 = 0\] are the basic variables.

Check:

\[\text{Constraint 1: } x_1 + x_3 - x_4 - x_5 = 1 + 1 = 2 \checkmark\]

\[\text{Constraint 2: } 2x_1 - x_2 = 2 - 1 = 2 \checkmark\]

\[\text{Constraint 3: } 8x_1 - 3x_2 + 2x_3 - 2x_4 = 8 + 2 = 10 \checkmark\]

\[\text{Objective: } 2x_1 + x_2 + 5x_3 - 5x_4 = 7\]
Now we begin phase 2. We are maximizing

\[-2x_1 - x_2 - 5x_3 + 5x_4\]

The basic variables are \(x_1, x_3,\) and \(x_5\). So we need to write the objective function in terms of the non-basic variables \(x_2\) & \(x_4\). We use the 3 eqns in the last tableau:

\[e_1: \quad \frac{1}{2}x_2 + x_3 - x_4 = 1 \quad \rightarrow \quad x_3 = -\frac{1}{2}x_2 + x_4 + 1\]
\[e_2: \quad x_1 - \frac{1}{2}x_2 = 1 \quad \rightarrow \quad x_1 = \frac{1}{2}x_2 + 1\]
\[e_3: \quad x_5 = 0\]

Objective function:

\[=-2\left(\frac{1}{2}x_2 + 1\right) - x_2 - 5\left(-\frac{1}{2}x_2 + x_4 + 1\right) + 5x_4\]
\[=-x_2 - 2 - x_2 + 5x_2 - 5x_4 - 5 + 5x_4\]
\[=3x_2 - 7\]

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_2)</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>(x_1)</td>
<td>1</td>
<td>-(\frac{1}{2})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(x_5)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\(x_2\) entering, \(x_3\) departing
<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_2$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>$X_1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$X_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0</td>
<td>6</td>
<td>-6</td>
<td>0</td>
</tr>
</tbody>
</table>

$x_4$ entering, no departing variable. The problem is unbounded $\Rightarrow$ no finite solution!

Note: had the constraints been $x_1, x_2, x_3 \geq 0$ then the optimal solution would have been $x_1=1, x_2=0, x_3=1$ with max. value $= 7.$
3. Consider the primal problem:

Minimize \( 4x_1 + 2x_2 + x_3 \) subject to

\[
\begin{align*}
2x_1 & + x_2 + x_3 \geq 1 \\
x_2 & - x_3 = 5 \\
x_1 & + x_2 - 2x_3 \leq 8
\end{align*}
\]

where \( x_1 \geq 0, x_2 \geq 0, \) and \( x_3 \) is unrestricted.

a. (3 points) State the dual problem

I know how to find the dual of

\[
\begin{align*}
\text{maximize } & \quad \tilde{c}^T \tilde{x} \\
\text{subject to } & \quad A \tilde{x} \leq \tilde{b} \\
& \quad \tilde{x} \geq 0
\end{align*}
\]

So I write the problem in that form:

I replace \( x_3 \) with \( x_3 - x_4 \) to do this.

\underline{primal:} \quad \text{maximize } -4x_1 - 2x_2 - x_3 + x_4

\underline{subject to:}

\[
\begin{align*}
-2x_1 & - x_2 - x_3 + x_4 \leq -1 \\
x_2 & - x_3 + x_4 = 5 \\
x_1 & + x_2 - 2x_3 + 2x_4 \leq 8
\end{align*}
\]

\( x_1, x_2, x_3, x_4 \geq 0 \)

I break the 2nd constraint into 2 inequalities:

\underline{primal:} \quad \text{maximize } -4x_1 - 2x_2 - x_3 + x_4

\underline{subject to:}

\[
\begin{align*}
-2x_1 & - x_2 - x_3 + x_4 \leq -1 \\
x_2 & - x_3 + x_4 \leq 5 \\
-x_2 & + x_3 - x_4 \leq -5 \\
x_1 & + x_2 - 2x_3 + 2x_4 \leq 8
\end{align*}
\]

\( x_1, x_2, x_3, x_4 \geq 0 \)
\[ \text{dual: } \text{minimize} \quad -w_1 + 5w_2 - 5w_3 + 8w_4 \]
\[ \text{subject to} \]
\[ -2w_1 + w_4 \geq 4 \]
\[ -w_1 + w_2 - w_3 + w_4 \geq 2 \]
\[ -w_1 - w_2 + w_3 - 2w_4 \geq -1 \]
\[ w_1 + w_2 - w_3 + 2w_4 \geq 1 \]
\[ w_1, w_2, w_3, w_4 \geq 0 \]

The 3rd & 4th constraints become an equality, and the variables \( w_2, w_3 \) always appear as opposite pairs, so I introduce \( u = -w_2 + w_3 \)

\[ \text{dual: } \text{minimize} \quad -w_1 - 5u + 8w_4 \]
\[ \text{subject to} \]
\[ -2w_1 + w_4 \geq 4 \]
\[ -w_1 + u + w_4 \geq 2 \]
\[ -w_1 + u - 2w_4 = -1 \]
\[ w_1, w_4 \geq 0, \quad u \text{ unrestricted} \]

Writing it all in terms of \( w_1, w_2, w_3 \):

\[ \text{dual: } \text{maximize} \quad w_1 + 5w_2 - 8w_3 \]
\[ \text{subject to} \]
\[ -2w_1 + w_3 \geq 4 \]
\[ -w_1 - w_2 + w_3 \geq 2 \]
\[ -w_1 + w_2 - 2w_3 = -1 \]
\[ w_1, w_3 \geq 0, \quad w_2 \text{ unrestricted} \]
b. (12 points) An optimal solution of the primal problem is \( x_1 = 0, x_2 = 3, x_3 = -2 \). Use this and complementary slackness to solve the dual problem.

**primal**: objective value = \( 4.0 + 2.3 + (-2) = 4 \)

**test for slackness**:

**const 1**: \( 2.0 + 3 + (-2) = 1 \geq 1 \) no slack

**const 2**: \( 3 - (-2) = 5 = 5 \) no slack (obviously)

**const 3**: \( 0 + 3 - 2(-2) = 7 < 8 \) slack.

This tells us that the slack variables of the primal problem are

\[ x_1' = 0 \quad x_2' = 0 \quad x_3' > 0 \]

since

\[ x_1', w_1 = 0 \]
\[ x_2', w_2 = 0 \] by compl. slackness, this gives us
\[ x_3', w_3 = 0 \]

\[ W_3 = 0 \]

On the other hand, complementary slackness also tells us

\[ x_1', w_1' = 0 \]
\[ x_2', w_2' = 0 \Rightarrow w_2' = 0 \]
\[ x_3', w_3' = 0 \Rightarrow w_3' = 0 \]

no slack in 2nd & 3rd constraints of dual problem.
so as the optimal solution of dual problem, we know

\[ w_1 + 5w_2 - 8w_3 = 4 \quad \text{← obj. function} \]

\[ w_3 = 0 \quad \text{← since slack in 3rd constr. of primal} \]

\[-w_1 - w_2 + w_3 = -2 \quad \text{← since } x_2 \neq 0 \]

\[-w_1 + w_2 - 2w_3 = -1 \quad \text{← since } x_3 \neq 0 \text{ and since it was an equality all along.} \]

\[-w_1 - w_2 = -2 \quad \Rightarrow -2w_1 = -3 \]

\[ \Rightarrow w_1 = \frac{3}{2} \]

\[ w_3 = 0 \]

\[-w_1 + w_2 = -1 \Rightarrow -\frac{3}{2} + w_2 = -1 \]

\[ \Rightarrow w_2 = \frac{1}{2} \]

check this is consistent with the first equation,

\[ \frac{3}{2} + 5\left(\frac{1}{2}\right) - 8(0) = 4 \]

\[ \checkmark \]

solution of dual problem:

\[ w_1 = \frac{3}{2} \]

\[ w_2 = \frac{1}{2} \]

\[ w_3 = 0 \]
4. (10 points)

Lyosha, Yael, Yi Li, and Vinh have been hired by a restaurant. The manager needs to assign four jobs: dishwasher, cook, busboy, and waiter. Yi Li insists on $7/hour to be a busboy or a dishwasher, $10/hour to be a waiter, and $11/hour to be a cook. Lyosha has more experience than Yi Li, so he is demanding $2 more per hour than she does, with one exception. He loves to cook, so he's willing to do that for $10/hour. Yael doesn't care what job she's given; she just wants $9/hour. Vinh has the least experience and so would accept $1 less per hour than Yi Li, with one exception: He has delicate hands and would wash dishes only for $12/hour.

The manager wants to assign the jobs in a way that would spend the least possible amount on wages. Who should get what job? How much per hour will the restaurant be paying for these four workers?

*In case you've never worked in a restaurant: busboys and waiters may be male or female.*

\[
C = \begin{pmatrix}
9 & 10 & 9 & 12 \\
9 & 9 & 9 & 9 \\
7 & 11 & 7 & 10 \\
12 & 10 & 6 & 9 \\
\end{pmatrix}
\]

Lyosha

Yael

Yi Li

Vinh

**step 1: subtract minimum from each row**

\[
C' = \begin{pmatrix}
0 & 1 & 0 & 3 \\
0 & 0 & 0 & 0 \\
0 & 4 & 0 & 3 \\
6 & 4 & 0 & 3 \\
\end{pmatrix}
\]

**step since there's a 0 in each row's column.**
\[ C' = \begin{pmatrix} 0^* & 1 & 0 & 3 \\ 0 & 0^* & 0 & 0 \\ 0 & 4 & 0^* & 3 \\ 6 & 4 & 0 & 3 \end{pmatrix} \]

4th row has no assigned 0.

\[(4, 3) \ 0^* \]
\[(3, 3) \ 0^* \]
\[(3, 1) \ 0 \]
\[(1, 1) \ 0^* \]

Shipping won't help.

As 1 & 3 are necessary.

Row 2 is necessary.

\[
\begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 3 \\ 6 & 4 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 \\ 6 & 3 & 0 & 2 \end{pmatrix}
\]

Assign 0's:

\[
\begin{pmatrix} 0^* & 0 & 0 & 2 \\ 1 & 0^* & 1 & 0 \\ 0 & 3 & 0^* & 2 \\ 6 & 3 & 0 & 2 \end{pmatrix}
\]

\[(4, 3) \ 0 \]
\[(3, 3) \ 0^* \]
\[(3, 1) \ 0 \]
\[(1, 1) \ 0^* \]
\[(1, 2) \ 0 \]
\[(2, 2) \ 0^* \]
\[(2, 4) \ 0 \]

Skip! and done

Lyosa = cook
Yael = washer
Yi Li = dish washer
Vinh = bus boy

Cost = 10 + 9 + 7 + 6 = 32 $/hour
5. (10 points) Solve the assignment problem with the given cost matrix. Show your work. Also, give the cost of the optimal solution.

Do not solve the problem by inspection! At each step, say what you're doing and make it clear to the grader that you're using the Hungarian algorithm of §5.2.

\[
C = \begin{pmatrix}
3 & 2 & 7 & 4 & 8 \\
5 & 4 & 3 & 8 & 5 \\
3 & 7 & 9 & 1 & 2 \\
4 & 2 & 6 & 5 & 7 \\
2 & 8 & 4 & 6 & 6
\end{pmatrix}
\]

\[
C' = \begin{pmatrix}
1 & 0 & 5 & 2 & 6 \\
2 & 1 & 0 & 5 & 2 \\
2 & 6 & 8 & 0 & 1 \\
2 & 0 & 4 & 3 & 5 \\
0 & 6 & 2 & 4 & 4
\end{pmatrix}
\]

subtract min from all rows.

\[
C' = \begin{pmatrix}
1 & 0 & 5 & 2 & 5 \\
2 & 1 & 0 & 5 & 1 \\
2 & 6 & 8 & 0 & 0 \\
2 & 0 & 4 & 3 & 4 \\
0 & 6 & 2 & 4 & 3
\end{pmatrix}
\]

subtract min from all columns

try assigning by the rule

\[
\begin{pmatrix}
1 & 0 & 5 & 2 & 5 \\
2 & 1 & 0 & 5 & 1 \\
2 & 6 & 8 & 0 & 0 \\
2 & 0 & 4 & 3 & 4 \\
0 & 6 & 2 & 4 & 3
\end{pmatrix}
\]

4th row has no assigned 0. Look for a change of assignments. (4,2) 0
(1,2) 0*

none poss. col 2 is nec.

\[
\text{rows 2, 3, 5 are nec.}
\]
$\begin{array}{c}
1 & 0 & 5 & 2 & 5 \\
2 & 1 & 0 & 5 & 1 \\
2 & 6 & 8 & 0 & 0 \\
2 & 0 & 4 & 3 & 4 \\
o & 6 & 2 & 4 & 3
\end{array}$

$c' = \begin{array}{c}
0^* & 0 & 4 & 1 & 4 \\
2 & 2 & 0^* & 5 & 1 \\
2 & 7 & 8 & 0^* & 0 \\
1 & 0^* & 3 & 2 & 3 \\
o & 7 & 2 & 4 & 3
\end{array}$

try assigning 0's...

try permuting them...

(5,1) 0
(1,1) 0^*  \Rightarrow \text{cols 1 & 2 are nec.}
(1,2) 0
(4,2) 0^*  \Rightarrow \text{rows 2 & 3 are nec.}

$\begin{array}{c}
0 & 0 & 4 & 1 & 4 \\
2 & 2 & 0^* & 5 & 1 \\
2 & 7 & 8 & 0^* & 0 \\
1 & 0^* & 3 & 2 & 3 \\
o & 7 & 2 & 4 & 3
\end{array}$

$\begin{array}{c}
0 & 0 & 3 & 0 & 3 \\
3 & 3 & 0^* & 5 & 1 \\
3 & 8 & 8 & 0^* & 0 \\
1 & 0^* & 2 & 1 & 2 \\
o & 7 & 1 & 3 & 2
\end{array}$

try permuting...

(5,1) 0
(1,1) 0*
(1,2) 0
(3,4) 0*
(3,5) 0

\text{Dip! } c' = \begin{array}{c}
0 & 0 & 3 & 0^* & 3 \\
3 & 3 & 0^* & 5 & 1 \\
3 & 8 & 8 & 0^* & 0 \\
1 & 0^* & 2 & 1 & 2 \\
o & 7 & 1 & 3 & 2
\end{array}$

$X_{14}=1, \ X_{23}=1, \ X_{35}=1, \ X_{42}=1, \ X_{51}=1$

$202 = 4 + 3 + 2 + 2 + 2 = 13$
6. (10 points) Consider the following transportation problem:

Minimize $2x_{11} + 2x_{12} + x_{13} + x_{21} + x_{22} + 3x_{23} + 2x_{31} + 2x_{32} + x_{33}$

Subject to

\[
\begin{align*}
x_{11} + x_{12} + x_{13} & = 25 \\
x_{21} + x_{22} + x_{23} & = 25 \\
x_{31} + x_{32} + x_{33} & = 10 \\
x_{11} + x_{21} + x_{31} & = 10 \\
x_{12} + x_{22} + x_{32} & = 20 \\
x_{13} + x_{23} + x_{33} & = 30 \\
\end{align*}
\]

where $x_{ij} \geq 0$ for $1 \leq i \leq 3, 1 \leq j \leq 3$.

Find an optimal solution. What is its cost?

\[
\begin{array}{|c|c|c|c|}
\hline
 & 2 & 3 & 4 \\
\hline
1 & 0 & 0 & 25 \\
\hline
2 & 1 & 2 & 20 \\
\hline
3 & 1 & 2 & 10 \\
\hline
\end{array}
\]

$x_{13} = 25, x_{21} = 5, x_{22} = 20, x_{31} = 5, x_{23} = 5$

$\text{Cost} = 25 + 5 + 20 + 10 + 5 = 65$

Is it optimal?

\[
\begin{align*}
x_{12} & : v_1 + w_3 = 1 \\
x_{21} & : v_2 = -1 \\
x_{22} & : v_2 + w_2 + 1 \\
x_{31} & : v_3 + w_1 = 2 \\
x_{33} & : v_3 + w_3 = 1 \\
\end{align*}
\]

\[
\begin{align*}
v_1 & = 0 \\
v_2 & = -1 \\
v_3 & = 0 \\
w_1 & = 2 \\
w_2 & = 2 \\
w_3 & = 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{obj. row} & : v_1 + w_1 - c_{11} = 0 + 2 - 2 = 0 \\
v_1 + w_2 - c_{12} = 0 + 2 - 2 = 0 \\
v_2 + w_3 - c_{23} = -1 + 1 - 3 = -3 \\
v_3 + w_2 - c_{32} = 0 + 2 - 2 = 0 \\
\end{align*}
\]

\(\text{None are > 0, so optimal.}\)
7. Consider the following transportation problem:

Minimize \( 2x_{11} + 1x_{12} + 3x_{13} + 2x_{21} + 3x_{22} + 2x_{23} + 4x_{31} + x_{32} + 3x_{33} \)

Subject to

\[
\begin{align*}
   x_{11} + x_{12} + x_{13} &= 10 \\
   x_{21} + x_{22} + x_{23} &= 20 \\
   x_{31} + x_{32} + x_{33} &= 25 \\
   x_{11} + x_{21} + x_{31} &= 15 \\
   x_{12} + x_{22} + x_{32} &= 20 \\
   x_{13} + x_{23} + x_{33} &= 20
\end{align*}
\]

where \( x_{ij} \geq 0 \) for \( 1 \leq i \leq 3, 1 \leq j \leq 3 \).

At one point in the minimization process, the basic variables are \( x_{11}, x_{21}, x_{22}, x_{31}, \) and \( x_{33} \).

a. (2 points) Fill in the transportation tableau below and compute the current value of the objective function.

\[
\begin{array}{c|cccc|c}
   & 1 & 2 & 3 & \text{Total} \\
\hline
   1 & 10 & 0 & 0 & 10 \\
   2 & 0 & 20 & 2 & 22 \\
   3 & 5 & 0 & 20 & 25 \\
\hline
   \text{Total} & 15 & 20 & 20 & \text{160}
\end{array}
\]

\[
\text{Cost} = 2 \cdot 10 + 3 \cdot 20 + 4 \cdot 5 + 3 \cdot 20 \\
= 20 + 60 + 20 + 60 \\
= 160
\]
b. (5 points) Complete the following simplex tableau

<table>
<thead>
<tr>
<th></th>
<th>$x_{11}$</th>
<th>$x_{12}$</th>
<th>$x_{13}$</th>
<th>$x_{21}$</th>
<th>$x_{22}$</th>
<th>$x_{23}$</th>
<th>$x_{31}$</th>
<th>$x_{32}$</th>
<th>$x_{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{11}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>$x_{21}$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_{22}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>$x_{31}$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$x_{32}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>120</td>
</tr>
<tr>
<td>$x_{33}$</td>
<td>0</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>160</td>
</tr>
</tbody>
</table>

**$X_{12}$ entering:**

\[
\begin{array}{c|c|c}
-1 & +1 \\
+1 & -1 \\
\end{array}
\]

Cost: \(1 - 2 + 2 - 3 = -2\)

**$X_{13}$ entering:**

\[
\begin{array}{c|c|c}
-1 & +1 \\
+1 & -1 \\
\end{array}
\]

Cost: \(3 - 2 + 4 - 3 = 2\)

**$X_{23}$ entering:**

\[
\begin{array}{c|c|c}
-1 & +1 \\
+1 & -1 \\
\end{array}
\]

Cost: \(2 - 2 + 4 - 3 = 1\)

**$X_{32}$ entering:**

\[
\begin{array}{c|c|c}
+1 & -1 \\
-1 & +1 \\
\end{array}
\]

Cost: \(1 - 3 + 2 - 4 = -4\)
c. (3 points) Choose an entering variable. What is the departing variable? Fill in the resulting transportation tableau below and find the new value of the objective function.

If $x_{12}$ enters then $x_{11}$ departs

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>25</td>
<td>25</td>
</tr>
</tbody>
</table>

15  20  20

$C_{04} = 160 - 10 \cdot 2 = 140$

If $x_{32}$ enters then $x_{31}$ departs

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>25</td>
<td>25</td>
</tr>
</tbody>
</table>

15  20  20

$C_{04} = 160 - 5 \cdot 4 = 140$
8. Consider the following simplex tableau.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-3</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>$x_5$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>$x_8$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>-1</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>$-2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>9</td>
</tr>
</tbody>
</table>

a. (1 point) What are the basic variables?

$x_3$ $x_2$ $x_5$ $x_8$

b. (1 point) What are the nonbasic variables?

$x_1$ $x_4$ $x_6$ $x_7$

c. (6 points) For each nonbasic variable, fill in the blanks in the following: "If ____ were to be an entering variable then the departing variable(s) would be ____ and the new value of the objective function would be ____." If you find this question confusing, look at parts d and e of this problem and see if that helps.

if $x_1$ enters then $x_5$ departs and
   the new value of obj. function would be 11

if $x_4$ enters then either $x_2$ or $x_8$ could depart.
   the new value of the obj. function would be 9

if $x_6$ enters then either $x_3$ or $x_5$ could
   depart. The new value of the obj. function
   would be 8

if $x_7$ enters then $x_8$ departs. The new
   value of the objective function would be 9
d. (1 point) If the simplex tableau arose as part of a minimization problem, what would you choose as the entering variable?

\[ x_0 \text{ entering.} \]

e. (1 point) If the simplex tableau arose as part of a maximization problem, what would you choose as the entering variable?

\[ x_1 \text{ entering.} \]