Vector fields

Exercise 1. Consider the smooth vector field \( v = x^k \frac{\partial}{\partial x}, k \geq 0 \) on the real line \( X = \mathbb{R} \). The flow \( \Phi_v(x, t) \) of the vector field \( v \) for time \( t \) starting at \( x \in X \) is defined for \( (x, t) \) in an open subset \( U \subset X \times \mathbb{R} \). Determine this open set precisely for each \( k \).

Exercise 2. Let \( v \) be a vector field on the manifold \( M \), and suppose it vanishes at the point \( p \in M \). In coordinates \((x^1, \ldots, x^n)\) centered at \( p \), we may write
\[
v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i},
\]
and the following expression defines an endomorphism of \( T_p M \):
\[
d_p v = \sum_{i=1}^{n} (dv^i) \big|_p \otimes \frac{\partial}{\partial x^i} \big|_p \in T^*_p M \otimes T_p M.
\]
Prove that \( d_p v \) does not depend on the choice of coordinates centered at \( p \).

Exercise 3. Let \( v \) be a vector field on \( M = \mathbb{R}^2 \) with an isolated zero at the origin. For a sufficiently small circle \( \gamma(t) = \varepsilon e^{it} \), the normalized vector field
\[
\sigma(t) = \frac{v(\gamma(t))}{|v(\gamma(t))|}
\]
defines a map \( S^1 \to S^1 \). The winding number of this map is called the index of the vector field at the origin.

1. Provide an explicit family of vector fields \( v_k \) on the plane with index \( k \) at the origin for \( k \in \mathbb{Z} \).

2. Given a continuous family \( v_t \) of vector fields on \( \mathbb{R}^2 \) parametrized by \( t \in \mathbb{R} \), such that \( v_t \) always has a single zero in the unit disc at the origin, prove that the index remains constant in the family. [This requires a basic understanding of what the fundamental group is, in particular the fact \( \pi_1(S^1) = \mathbb{Z} \).]

3. Suppose that the vector field \( v \) on \( \mathbb{R}^2 \) is nonvanishing on the unit circle \( \gamma(t) = e^{it} \), and suppose that the winding number of the map (1) is nonzero. Prove that \( v \) must have a zero somewhere in the unit disc.

4. Use the above to prove that \( S^2 \) cannot have a nowhere-vanishing vector field. Use the description of \( S^2 \) and its tangent bundle in terms of a pair of stereographic charts.
Transversality

Vector subspaces $U, V$ of $W$ are transverse when $U + V = W$. Two submanifolds $K, L$ of the manifold $M$ intersect transversally if at each point $p \in K \cap L$, the tangent spaces $T_pK$ and $T_pL$ are transverse in $T_p M$.

**Exercise 4.** Prove that if the submanifolds $K, L$ of $M$ intersect transversally, then $K \cap L$ is also a submanifold. Also, determine the dimension of the intersection.

For each $k = 0, 1, \ldots$ give an example of two transversally intersecting submanifolds $L, K$ of $S^1 \times S^1$ which intersect in exactly $k$ points.

**Exercise 5.** Sard’s theorem states that for any smooth map, the set of critical values has measure zero in the codomain. In other words, the regular values are dense. Recall that for a point $y$ in the codomain of $f$ to be regular, each point in the preimage $f^{-1}(y)$ must be regular, i.e. have surjective derivative. (Important point: if $f^{-1}(y)$ is empty, then $y$ is regular!).

1. If $f : M \to M$ is a smooth map from a compact manifold to itself, prove that there must be a point $y \in M$ with $f^{-1}(y)$ finite.

2. If $f : M \to S^n$ is a smooth map and $\dim M < n$, prove that $f$ is smoothly homotopic to a constant map. ‘Smoothly homotopic’ in this case would mean that you have a smooth map

   $$F : M \times [0, 1] \to S^n$$

   with $F(-, 0) = f(-)$ and $F(-, 1)$ being a constant map.

**Exercise 6.** We say that a smooth map $f : K \to M$ is transverse to the submanifold $L \subset M$ if $Df(T_pK) + T_{f(p)}L = T_{f(p)}M$ for all $p \in f^{-1}(L)$. If $f$ were an embedding of the submanifold $K$, we would recover the usual notion of transversality.

Let $S$ be another manifold (think of it as a parameter space) and suppose that $F : K \times S \to M$ is a smooth map which is transverse to $L$. We would like to know if the individual maps $F(-, s) : K \to M$, where $s$ is fixed, are transverse to $L$.

1. Prove that $Q = F^{-1}(L)$ is a smooth submanifold of $K \times S$.

2. Let $\pi : Q \to S$ be the projection map. Prove that if $s$ is a regular value for $\pi$, then $F(-, s) : K \to M$ is transverse to $L$. Conclude that $F(-, s) : K \to M$ is transverse to $L$ for almost all $s$.

**Exercise 7.** Let $f$ be a smooth real-valued function on the compact manifold $M$ such that $df$ is transverse to the zero section, meaning that the image of the section $df \in \Gamma(M, T^*M)$ in $T^*M$ defines a submanifold which intersects the image of the zero section transversally. Prove that $f$ has finitely many critical points, at each of which its Hessian is nondegenerate.