2 Homology

We now turn to Homology, a functor which associates to a topological space $X$ a sequence of abelian groups $H_k(X)$. We will investigate several important related ideas:

- Homology, relative homology, axioms for homology, Mayer-Vietoris
- Cohomology, coefficients, Poincaré Duality
- Relation to de Rham cohomology (de Rham theorem)
- Applications

The basic idea of homology is quite simple, but it is a bit difficult to come up with a proper definition. In the definition of the homotopy group, we considered loops in $X$, considering loops which could be “filled in” by a disc to be trivial.

In homology, we wish to generalize this, considering loops to be trivial if they can be “filled in” by any surface; this then generalizes to arbitrary dimension as follows (let $X$ be a manifold for this informal discussion).

A $k$-dimensional chain is defined to be a $k$-dimensional submanifold with boundary $S \subset X$ with a chosen orientation $\sigma$ on $S$. A chain is called a cycle when its boundary is empty. Then the $k$th homology group is defined as the free abelian group generated by the $k$-cycles (where we identify $(S, \sigma)$ with $-(S, -\sigma)$), modulo those $k$-cycles which are boundaries of $k+1$-chains. Whenever we take the boundary of an oriented manifold, we choose the boundary orientation given by the outward pointing normal vector.

**Example 2.1.** Consider an oriented loop separating a genus 2 surface into two genus 1 punctured surfaces. This loop is nontrivial in the fundamental group, but is trivial in homology, i.e. it is homologous to zero.

**Example 2.2.** Consider two parallel oriented loops $L_1, L_2$ on $T^2$. Then we see that $L_1 - L_2 = 0$, i.e. $L_1$ is homologous to $L_2$.

**Example 2.3.** This definition of homology is not well-behaved: if we pick any embedded submanifold $S$ in a manifold and slightly deform it to $S'$ which still intersects $S$, then there may be no submanifold with $S \cup S'$ as its boundary. We want such deformations to be homologous, so we slightly relax our requirements: we allow the $k$-chains to be smooth maps $\iota: S \rightarrow M$ which needn’t be embeddings.

This definition is still problematic: it’s not clear what to do about non-smooth topological spaces, and also the definition seems to require knowledge of all possible manifolds mapping into $M$. We solve both problems by cutting $S$ into triangles (i.e. simplices) and focusing only on maps of simplices into $M$.

**Definition 11.** An $n$-simplex $[v_0, \cdots, v_n]$ is the convex hull of $n+1$ ordered points (called vertices) in $\mathbb{R}^n$ for which $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent.

The standard $n$-simplex is

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \forall i\},$$

and there is a canonical map $\Delta^n \rightarrow [v_0, \cdots, v_n]$ via

$$(t_0, \ldots, t_n) \mapsto \sum_i t_i v_i,$$

called barycentric coordinates on $[v_0, \cdots, v_n]$. A face of $[v_0, \cdots, v_n]$ is defined as the simplex obtained by deleting one of the $v_i$, we denote it $[v_0, \cdots, \check{v}_i, \cdots, v_n]$. The union of all faces is the boundary of the simplex, and its complement is called the interior, or the open simplex.
Definition 12. A $\Delta$-complex decomposition of a topological space $X$ is a collection of maps $\sigma_\alpha : \Delta^n \longrightarrow X$ ($n$ depending on $\alpha$) such that $\sigma_\alpha$ is injective on the open simplex $\Delta^n_\alpha$, every point is in the image of exactly one $\sigma_\alpha|\Delta^n_\alpha$, and each restriction of $\sigma_\alpha$ to a face of $\Delta^n(\alpha)$ coincides with one of the maps $\sigma_\beta$, under the canonical identification of $\Delta^{n-1}$ with the face (which preserves ordering). We also require the topology to be compatible: $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in the simplex for each $\alpha$.

It is easy to see that such a structure on $X$ actually expresses it as a cell complex.

Example 2.4. Give the standard decomposition of 2-dimensional compact manifolds.

We may now define the simplicial homology of a $\Delta$-complex $X$. We basically want to mod out cycles by boundaries, except now the chains will be made of linear combinations of the $n$-simplices which make up $X$.

Let $\Delta_n(X)$ be the free abelian group with basis the open $n$-simplices $e^n_\alpha = \sigma_\alpha(\Delta^n)$ of $X$. Elements $\sum_\alpha n_\alpha \sigma_\alpha \in \Delta_n(X)$ are called $n$-chains (finite sums).

Each $n$-simplex has a natural orientation based on its ordered vertices, and its boundary obtains a natural orientation from the outward-pointing normal vector field. Algebraically, this induced orientation is captured by the following formula (which captures the interior product by the outward normal vector to the $i$th face):

$$\partial[v_0, \ldots, v_n] = \sum_i (-1)^i[v_0, \ldots, \hat{v}_i, \ldots, v_n].$$

This allows us to define the boundary homomorphism:

Definition 13. The boundary homomorphism $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ is determined by

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{\Delta^n_i}.$$ 

This definition of boundary is clearly a triangulated version of the usual boundary of manifolds, and satisfies $\partial \circ \partial = 0$, i.e.

Lemma 2.5. The composition $\partial_{n-1} \circ \partial_n = 0$.

Proof.

$$\partial \partial [v_0, \ldots, v_n] = \sum_{j<i} (-1)^{i+j}[v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n] + \sum_{j>i} (-1)^{i+j-1}[v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]$$

the two displayed terms cancel.

Now we have produced an algebraic object: a chain complex (just as we saw in the case of the de Rham complex). Let $C_n$ be the abelian group $\Delta_n(X)$; then we get the simplicial chain complex:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_n} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

and the homology is defined as the simplicial homology

$$H^n(X) := \frac{Z_n}{B_n} = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$ 

Example 2.6. The circle is a $\Delta$-complex with one vertex and one 1-simplex. so $\Delta_0(S^1) = \Delta_1(S^1) = \mathbb{Z}$ and $\partial_1 = 0$ since $\partial e = v - v$. hence $H^0_\Delta(S^1) = \mathbb{Z} = H^1_\Delta(S^1)$ and $H^2_\Delta(S^1) = 0$ otherwise.

Example 2.7. For $T^2$ and Klein bottle: $\Delta_0 = \mathbb{Z}$, $\Delta_1 = \langle a, b, c \rangle$ and $\Delta_2 = \langle P, Q \rangle$. For $\mathbb{R}P^2$, same except $\Delta_0 = \mathbb{Z}^2$. 

20
Simplicial homology, while easy to calculate (at least by computer!), is not entirely satisfactory, mostly because it is so rigid - it is not clear, for example, that the groups do not depend on the triangulation. We therefore relax the definition and describe singular homology.

**Definition 14.** A singular $n$-simplex in a space $X$ is a continuous map $\sigma : \Delta^n \to X$. The free abelian group on the set of $n$-simplices is called $C_n(X)$, the group of $n$-chains.

There is a linear boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ given by

$$ \partial_n \sigma = \sum_i (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}, $$

where $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ is canonically identified with $\Delta^{n-1}$. The homology of the chain complex $(C_\bullet(X), \partial)$ is called the *singular homology* of $X$:

$$ H_n(X) := \frac{\ker \partial : C_n(X) \to C_{n-1}(X)}{\im \partial : C_{n+1}(X) \to C_n(X)}. $$