Exercise 1. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$(r, \phi, \theta) \mapsto (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi),$$

where $(r, \phi, \theta)$ are standard Cartesian coordinates on $\mathbb{R}^3$.

• Show that $\varphi$ is a diffeomorphism onto its image when restricted to $U = \{(r, \phi, \theta) : 0 < r, 0 < \phi < \pi, 0 < \theta < 2\pi\}$.

• Compute $\varphi^*dx, \varphi^*dy, \varphi^*dz$ where $(x, y, z)$ are Cartesian coordinates for $\mathbb{R}^3$.

• Compute $\varphi^*(dx \wedge dy \wedge dz)$.

• For any vector field $X$, define $\iota_X$ to be the unique degree $-1$ (i.e. it reduces degree by 1) derivation (i.e. $\iota_X(\alpha \wedge \beta) = \iota_X(\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \iota_X(\beta)$) of the algebra of differential forms such that $\iota_X(f) = 0$ and $\iota_X df = X(f)$ for $f \in \Omega^0(M)$. Compute the integral

$$\int_{S^2} \iota_X(dx \wedge dy \wedge dz),$$

for the vector field $X = \varphi^* \frac{\partial}{\partial r}$, where $S^2_r$ is the sphere of radius $r$.

Exercise 2. Let $z$ be the standard complex coordinate on $\mathbb{C}$, i.e. $z = x + iy$, and form the complex differential form $dz$ $z$. Where is this well-defined? Decompose it into real and imaginary parts. Are these closed forms? Compute the integrals

$$\int_{S^1} \iota^* \mu,$$

for $\iota : S^1 \to \mathbb{C}$ the standard inclusion and $\mu$ the real or imaginary part of $\frac{dz}{z}$. What consequence does Stokes’ theorem have in this context, for each part of $\frac{dz}{z}$?

Exercise 3. Let $f_0 : M \to N$, $f_1 : M \to N$ be smoothly homotopic maps, i.e. there exists a smooth map

$$h : M \times \mathbb{R} \to N$$

such that $h(x, i) = f_i(x)$ for $i = 0, 1$. Then show that $(f_1^* - f_0^*)\alpha$ is exact when $\alpha$ is closed.

Exercise 4. Let $\{U_i\}, i = 1, \ldots, N$ be a finite cover of a compact, oriented $n$-manifold $M$, and let $\alpha \in \Omega^n(M)$. Express $\int_M \alpha$ in terms of the integrals

$$\int_{U_{i_1} \cap \cdots \cap U_{i_k}} \alpha$$

for $k$ ranging from 1 to $N$.

Exercise 5. Compute all these de Rham cohomology groups for all degrees.

• What is the de Rham cohomology of $\mathbb{R}^3 - \{p_1 \cup \cdots \cup p_k\}$ where $p_i$ are a collection of $k$ distinct points?

• What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup \cdots \cup l_m\}$ where $l_i$ are a collection of $m$ non-intersecting lines?

• What is the de Rham cohomology of $\mathbb{R}^3 - \{p_1 \cup \cdots \cup p_k \cup l_1 \cup \cdots \cup l_m\}$, assuming no $p_i$ lies on a $l_j$, $\{p_i\}$ are distinct, and $\{l_j\}$ are non-intersecting?

• What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup l_2\}$, assuming that $l_1$ intersects $l_2$ in exactly one point?

• What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup \cdots \cup l_m\}$, assuming that all the lines intersect the origin but are distinct?
• What is the de Rham cohomology of \( \mathbb{R}^3 - \{l_1 \cup l_2 \cup l_3\} \), assuming the lines intersect in exactly three distinct points?

• What is the de Rham cohomology of \( \mathbb{R}^n - \{X_i\} \), where \( X_i \) is an \( i \)-dimensional linear subspace?

Note: The preceding question is similar to John Nash’s blackboard question in the movie “A Beautiful Mind”.

Exercise 6. Let \( N = T^*M \) be the total space of the cotangent bundle of a smooth manifold \( M \), and let \( \pi : N \longrightarrow M \) be the usual bundle projection. We now describe a natural 1-form \( \theta \in \Omega^1(N) \). At each point \( p = (x,\xi) \in N \) (here \( x \in M \) is a point and \( \xi \in T^*_xM \) is a covector at \( x \)), the 1-form takes the following value on a vector \( V \in T_pN \):

\[
\theta(V) = \xi(\pi_*V).
\]

i) Choosing coordinates \((x^1,\ldots,x^n)\) for an open set \( U \) containing \( x \) and using coordinates \((x^1,\ldots,x^n,\xi_1,\ldots,\xi_n)\) to represent the points \( (x \in U; \xi = \sum \xi_i dx^i) \in T^*U \), write the coordinate expression of \( \theta \) and verify that it is smooth.

ii) Compute \( \omega = d\theta \in \Omega^2(N) \). View the result as a smooth family of skew-symmetric 2-forms on \( N \). Compute the rank of this 2-form at the point \( p = (x,\xi) \) (i.e. if we write \( \omega = \sum \omega_{ij} dx^i \wedge dx^j \), the rank is the rank of the matrix \( \omega_{ij} \)).

iii) Let \( \mu \in \Omega^1(M) \) be a 1-form on \( M \), and view it as a smooth section \( \mu : M \longrightarrow T^*M \) of the cotangent bundle. Therefore it defines a smooth map \( \mu : M \longrightarrow N \). Compute the pullbacks \( \mu^*(\theta) \in \Omega^1(M) \) and \( \mu^*(\omega) \in \Omega^2(M) \) as a function of \( \mu \).

iv) Just as a natural 1-form \( \theta \) was defined on \( T^*M \), define a natural \( k \)-form \( \theta \in \Omega^k(N_k) \) on the total space of the bundle \( N_k = \wedge^k T^*M \). If \( \mu \in \Omega^k(M) \), view it as a smooth map \( \mu : M \longrightarrow N_k \) and compute \( \mu^*(d\theta) \). Does all this work for \( k = 0 \)?