Problem #1: [2+1+1+1=5 points]
Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be the map
$$F(x, y, z) = (yz, xz, xy)$$
a) For $k = 0, 1, 2, 3$, determine the set of all $p \in \mathbb{R}^3$ where $\text{rank}_p(F) = k$.
b) Describe the set of all singular values $q = (a, b, c)$.
c) Describe the level sets $F^{-1}(q)$ for a regular value $q = (a, b, c)$.
d) Describe the level set $F^{-1}(q)$ for a singular value $q = (a, b, c)$.

Problem #2: [2+2=4 points]
Consider the map
$$F: \mathbb{R}^3 \to \mathbb{R}^2, \ (x, y, z) \mapsto (z^2 - x^2 - y^2, z - x).$$
a) Find all points $p \in \mathbb{R}^3$ where $F$ fails to have maximal rank. What is the rank at those points?
b) Find all singular values $a \in \mathbb{R}^2$ of the map $F$, and describe (e.g., sketch) the corresponding level sets $F^{-1}(a)$.

Problem #3: [1+1+2+2=6 points]
Let $M \subset \mathbb{R}^3$ be the following 2-torus,
$$M = \{(x, y, z) \in \mathbb{R}^3 | (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\},$$
and let $F \in C^\infty(M, \mathbb{R})$ the function $F(x, y, z) \mapsto y$.
a) Show that the subset of $M$ where $1 < x^2 + y^2 < 9$ and $z > 0$ is described as the graph of a smooth function $z = f(x, y)$. Conclude that on this region, one can use $x, y$ as coordinates.
b) Describe similar ‘largest’ regions where $x, z$ (or $y, z$) can serve as coordinates.
c) Sketch the level surfaces $F^{-1}(a)$ for the values $a = -3, -2, -1, 0, 1, 2, 3$. Which of these level sets don’t look like 1-dimensional submanifolds? (It may be best to sketch the torus in $x-z-y$-coordinates, with the $y$-axis pointing up.)
d) Find the critical points $p$ of $F$. What are the corresponding singular values $a = F(p) \in \mathbb{R}$? (It will suffice to do a calculation in $x-z$-coordinates – why?)

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Problem #4: [4+1=5 points]

Let $\text{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ be the space of real $n \times n$-matrices, and $\text{Sym}_n(\mathbb{R}) \cong \mathbb{R}^{n(n+1)/2}$ the subspace of symmetric matrices. For any matrix $A$, let $A^\top$ be its transpose. Consider the map

$$F : \text{Mat}_n(\mathbb{R}) \to \text{Sym}_n(\mathbb{R}), \quad A \mapsto A^\top A.$$ 

a) For any $A \in \text{Mat}_n(\mathbb{R})$, calculate the linear map $D_A F : \text{Mat}_n(\mathbb{R}) \to \text{Sym}_n(\mathbb{R}), \quad X \mapsto (D_A F)(X)$.

Show that the identity matrix $I \in \text{Sym}_n(\mathbb{R})$ is a regular value of $F$. (Hint: Consider $X$ of the form $AY$.)

b) Conclude that the orthogonal group $O(n) = \{A \in \text{Mat}_n(\mathbb{R}) \mid A^\top A = 1\}$ is a submanifold of $\text{Mat}_n(\mathbb{R})$. What is its dimension?

Extra question (Do not hand in.)

Let $(U_j, \phi_j)$, $j = 0, 12$, be the standard charts for $\mathbb{RP}(2)$. Consider the map

$$F : \mathbb{RP}(2) \to \mathbb{R}^3, \quad \(x^0 : x^1 : x^2\) \mapsto \frac{1}{||x||^2}(x^1 x^2, x^0 x^2, x^0 x^1)$$

where $||x||^2 = (x^0)^2 + (x^1)^2 + (x^2)^2$.

(a) Find all points $p \in \mathbb{RP}(2)$ such that $F(p) = 0$. How many such points are there?

(b) Compute the differential and the rank of the map

$$\mathbb{R}^2 \to \mathbb{R}^3, \quad (u, v) \mapsto F(\phi_0^{-1}(u, v)),$$

at any given point $(u, v)$.

(c) Find all points $p \in \mathbb{RP}(2)$ where $F$ fails to have maximal rank, and show that the rank at those points is 1.

(d) Given a non-zero point $0 \neq q = (a, b, c) \in \mathbb{R}^3$ in the image of $F$, show that there is a unique point $p \in \mathbb{RP}(2)$ such that $F(p) = q$.

The significance of these calculations is as follows: By part (d), the map $F$ is injective away from $F^{-1}(0)$, and by part (a) there are only finitely many points in $F^{-1}(0)$. Hence, if $f \in C^\infty(\mathbb{RP}(2))$ takes on different values at the points of $F^{-1}(0)$, then the map

$$(F, f) : \mathbb{RP}(2) \to \mathbb{R}^4$$

will be injective. If furthermore the set of critical points for $f$ doesn’t meet the set of points where $\text{rank}_p(F) < 2$, then the map $(F, f)$ will be an injective immersion, realizing $\mathbb{RP}(2)$ as a submanifold of $\mathbb{R}^4$. For instance,

$$f(x^0 : x^1 : x^2) = \frac{1}{||x||^2} \left( (x^0)^2 + 2(x^1)^2 + 3(x^2)^2 \right)$$

has these properties.