Problem #1: [3 points]
Give an example of an invertible smooth map whose inverse is not smooth.

Problem #2: [3+3+1 points] For this problem, you will need to review the ‘standard atlas’ \((U_I, \phi_I)\) for the Grassmannian; see the lecture notes.

a) Show that the map \(\text{Gr}(k,n) \rightarrow \text{Gr}(n-k,n)\), taking a subspace \(E\) to the orthogonal subspace \(E^\perp\), is a diffeomorphism.

b) For \(k \leq n\), let \(\text{St}(k,n) \subset L(\mathbb{R}^k,\mathbb{R}^n)\) be the ‘Stiefel manifold’ of linear maps of rank \(k\). (It is a manifold, being an open subset of \(L(\mathbb{R}^k,\mathbb{R}^n) \cong \mathbb{R}^{kn}\).) Let
\[
F: \text{St}(k,n) \rightarrow \text{Gr}(k,n)
\]
be the map taking such a linear map \(C: \mathbb{R}^k \rightarrow \mathbb{R}^n\) to its range:
\[
F(C) = C(\mathbb{R}^k) \subset \mathbb{R}^n.
\]
Show that the map \(F: \text{St}(2,4) \rightarrow \text{Gr}(2,4)\) is smooth. (You may want to think of \(C\) as a \(4 \times 2\)-matrix. When is \(F(C) \in U_I\)? If so, what is the linear map \(A_I = \phi_I(F(C)): \mathbb{R}^I \rightarrow \mathbb{R}^{I'}\)?) (If you prefer, you can also do the general case.)

c) Explain briefly how the map from b) generalizes the quotient map \(\mathbb{R}^{m+1}\setminus\{0\} \rightarrow \mathbb{R}P(m)\).

Problem #3: [4 points]
Consider the map \(F: \mathbb{C}P(1) \rightarrow \mathbb{C}P(1)\), described in homogeneous coordinates as
\[
F(z : 1) = (z + 1 : 1), \quad F(1 : 0) = (1 : 0).
\]
Is this map smooth? Justify your answer.

Problem #4: [4+2 points]
The following problem shows how to obtain \(S^3\) from two copies of \(\mathbb{R}^2 \times S^1\) by suitable ‘gluing’ along \((\mathbb{R}^2\setminus\{0\}) \times S^1\).

Viewing \(S^3\) as the unit sphere inside \(\mathbb{C}^2 = \mathbb{R}^4\), the Hopf map is the map
\[
F: S^3 \rightarrow \mathbb{C}P(1), \quad (z^0, z^1) \mapsto (z^0 : z^1).
\]
Let \((U_0, \phi_0), (U_1, \phi_1)\) be the standard atlas for \(\mathbb{C}P(1)\).

(a) What are the pre-images \(F^{-1}(U_0), F^{-1}(U_1), F^{-1}(U_0 \cap U_1)\)? Find explicit diffeomorphisms, for \(i = 0, 1,\)
\[
H_i: U_i \times S^1 \rightarrow F^{-1}(U_i),
\]
such that \(F(H_i(p, e^{i\theta})) = p\) for all \(p \in U_i\). (We regard \(S^1\) as the set of complex numbers of absolute value 1.)

(b) The two maps \(H_0, H_1\) restrict to diffeomorphisms \((U_0 \cap U_1) \times S^1 \rightarrow F^{-1}(U_0 \cap U_1)\). Calculate the diffeomorphism
\[
K: (U_0 \cap U_1) \times S^1 \rightarrow (U_0 \cap U_1) \times S^1
\]
such that \(H_1 = H_0 \circ K\).

Continued on back side.
Let $D^2 = \{ x \in \mathbb{R}^2 \mid ||x|| = 1 \}$ be the unit disk. It is an example of a “manifold with boundary”; in this case the boundary is $S^1$. The solid 2-torus $D^2 \times S^1$ is another example; here the boundary is $S^1 \times S^1$.

The 2-sphere $S^2$ is obtained from two copies of $D^2$ by gluing along the boundary. Using the last problem, think about how to obtain $S^3$ similarly from two copies of the solid 2-torus $D^2 \times S^1$ by gluing along the boundary. (You have to get the gluing right – otherwise you obtain $S^2 \times S^1$, or some other 3-dimensional manifold.) This is an example of a Heegard decomposition.

Note: “manifold with boundary” is a concept that we haven’t properly discussed (yet). You may want to look up the definition, or perhaps just use your intuition how this is defined.

Extra question (Do not hand in.)

Again, we regard $S^3$ as the unit sphere inside $\mathbb{C}^2 = \mathbb{R}^4$, and let $F: S^3 \to \mathbb{C}P(1)$ be the Hopf map. Identify $\mathbb{R}^3$ with the subspace of $\mathbb{C}^2$ consisting of $(z, w)$ with $\text{Im}(w) = 0$. Stereographic projection from the point $(0, i) \in S^3$ defines a diffeomorphism

$$\phi: S^3 \setminus \{(0, i)\} \to \mathbb{R}^3.$$ 

Work out explicitly the images of the circles $F^{-1}(p)$ for $p \in \mathbb{C}P(1)$ under the stereographic projection.

Puzzle (Just for fun. Do not hand this in – actually, it is not really a differential geometry question): Present $\mathbb{R}^3$ as a disjoint union of circles.