Appendix A
Topology of manifolds

A.1 Toplogical notions

A *topological space* is a set $X$ together with a collection of subsets $U \subseteq X$ called *open subsets*, with the following properties:

- $\emptyset, X$ are open.
- If $U, U'$ are open then $U \cap U'$ is open.
- For any collection $U_i$ of open subsets, the union $\bigcup_i U_i$ is open.

The collection of open subsets is called the *topology* of $X$. The space $\mathbb{R}^m$ has a standard topology given by the usual open subsets. Likewise, the open subsets of a manifold $M$ define a topology on $M$. For any set $X$, one has the *trivial topology* where the only open subsets are $\emptyset$ and $X$, and the *discrete topology* where every subset is considered open. An *open neighborhood* of a point $p$ is an open subset containing it. A topological space is called *Hausdorff* if any two distinct points have disjoint open neighborhoods.

Any subset $A \subseteq X$ has a *subspace topology*, with open sets the collection of all intersections $U \cap A$ such that $U \subseteq X$ is open. Given a surjective map $q : X \to Y$, the space $Y$ inherits a *quotient topology*, whose open sets are all $V \subseteq Y$ such that the pre-image $q^{-1}(V) = \{x \in X \mid q(x) \in V\}$ is open.

A subset $A$ is *closed* if its complement $X \setminus A$ is open. Dual to the statements for open sets, one has that the intersection of an arbitrary collection of closed sets is closed, and the union of two closed sets is closed. For any subset $A$, denote by $\overline{A}$ its *closure*, given as the smallest closed subset containing $A$.

A.2 Manifolds are second countable

A *basis* for the topology on $X$ is a collection $\mathcal{B} = \{U_a\}$ of open subsets of $X$ such that every $U$ is a union from sets from $\mathcal{B}$. A topological space is said to be *second countable* if its topology has a countable basis.
Proposition A.1. Manifolds are second countable.

Proof. The space $\mathbb{R}^m$ is second countable: A basis is given by the collection of all rational balls, by which we mean $\varepsilon$-balls $B_\varepsilon(x)$ such that $x \in \mathbb{Q}^m$ and $\varepsilon \in \mathbb{Q}_{>0}$. To check it is a basis, let $U \subseteq \mathbb{R}^m$ be open, and $p \in U$. Choose $\varepsilon \in \mathbb{Q}_{>0}$ such that $B_{2\varepsilon}(p) \subseteq U$. There exists a rational point $x \in \mathbb{Q}^m$ with $||x - p|| < \varepsilon$. This then satisfies $p \in B_{\varepsilon}(x) \subseteq U$. Since $p$ was arbitrary, this proves the claim. The same reasoning shows that for any open subset $U \subseteq \mathbb{R}^m$, the rational $\varepsilon$-balls that are contained in $U$ form a basis of the topology of $U$.

Given a manifold $M$, let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ be a countable atlas. Then the set of all $\phi_\alpha^{-1}(B_\varepsilon(x))$, where $B_\varepsilon(x)$ is a rational ball contained in $\phi_\alpha(U_\alpha)$, is a countable basis for the topology of $M$. Indeed, any open subset $U$ is a countable union over all $U \cap U_\alpha$, and each of these intersections is a countable union over all $\phi_\alpha^{-1}(B_\varepsilon(x))$ such that $B_\varepsilon(x)$ is a rational $\varepsilon$-ball contained in $U \cap U_\alpha$.

A collection $\{U_\alpha\}$ of open subsets of $X$ is called an open covering of $A \subseteq X$ if $A \subseteq \bigcup_\alpha U_\alpha$. Consider the case $A = X$. A refinement of an open cover $\{U_\alpha\}$ of $X$ is an open cover $\{V_\beta\}$ of $X$ such that each $V_\beta$ is contained in some $U_\alpha$. If $V_\beta = U_{f(\beta)}$ for all $\beta$, then the refinement is called a subcover.

A.3 Manifolds are paracompact

A topological space $X$ is called compact if every open cover of $X$ has a finite subcover. A topological space is called paracompact if every open cover $\{U_\alpha\}$ has a locally finite refinement $\{V_\beta\}$: that is, every point has an open neighborhood meeting only finitely many $V_\beta$’s.

Proposition A.2. Manifolds are paracompact.

Proof. Let $\{U_\alpha\}$ be an open cover of $M$. The first step is to construct a sequence of open subsets $W_1, W_2, \ldots$ such that

$$\bigcup W_i = M,$$

and each $W_i$ has compact closure with $\overline{W_i} \subseteq W_{i+1}$.

To this end, start with a a countable open cover $O_1, O_2, \ldots$ of $M$ such that each $O_i$ has compact closure $\overline{O_i}$. (We saw in the proof of the previous proposition how to construct such a cover.) Replacing $O_i$ with $O_1 \cup \cdots \cup O_i$ we may assume $O_1 \subseteq O_2 \subseteq \cdots$. Define $W_1, W_2, \ldots$ as a subsequence of $O_1, O_2, \ldots$, by inductively letting $W_i = O_{j(i)}$, with the smallest index $j(i)$ such that $\overline{W_{i-1}} \subseteq O_{j(i)}$ (the induction starts with $W_0 := \emptyset$).\footnote{If $M$ is compact, then the sequence $W_i$ may become independent of $i$ for large $i$.}
The compact subset $\overline{W}_{i+1}\setminus W_i$ is contained in the open set $W_{i+2}\setminus \overline{W}_{i-1}$. It is covered by the open sets $(W_{i+2}\setminus \overline{W}_{i-1}) \cap U_\alpha$, and by compactness it is already covered by finitely many of those intersections. Let $\mathcal{V}^{(i)}$ be this finite collection, and $\mathcal{V} = \bigcup \mathcal{V}^{(i)} = \{V_\beta\}$ the open cover given as the union over all $i$.

By construction, $\mathcal{V} = \{V_\beta\}$ is a countable cover, which is locally finite: Given $p \in M$, choose $i$ such that $p \in W_i$; by construction this $W_i$ can only meet $V_\beta$'s from $\mathcal{V}^{(k)}$ with $k \leq i$.

\begin{remark}
(Cf. Lang, page 35.) One can strengthen the result a bit, as follows: Given a cover $\{U_\alpha\}$, we can find a refinement to a cover $\{V_\beta\}$ such that each $V_\beta$ is the domain of a coordinate chart $(V_\beta, \psi_\beta)$, with the following extra properties, for some $0 < r < R$:

(i) $\psi_\beta(V_\beta) = B_r(0)$, and

(ii) $M$ is already covered by the smaller subsets $V_\beta' = \psi_\beta^{-1}(B_r(0))$.

To prove this, we change the second half of the proof as follows: For each $p \in \overline{W}_{i+1}\setminus W_i$ choose a coordinate chart $(V_p, \psi_p)$ such that $\psi_p(p) = 0$, $\psi_p(V_p) = B_R(0)$, and $V_p \subseteq (W_{i+2}\setminus \overline{W}_{i-1}) \cap U_\alpha$. Let $V_p' \subseteq V_p$ be the pre-image of $B_r(0)$. The $V_p$ cover $\overline{W}_{i+1}\setminus W_i$; let $\mathcal{V}^{(i)}$ be a finite subcover and proceed as before. This remark is useful for the construction of partitions of unity.

\section{Partitions of unity}

We will need the following result from multivariable calculus.

\begin{lemma}[Bump functions]
For all $0 < r < R$, there exists a function $f \in C^\infty(\mathbb{R}^n)$, with $f(x) = 0$ for $||x|| \geq R$ and $f(x) = 1$ for $||x|| \leq r$.
\end{lemma}

\begin{proof}
It suffices to prove the existence of a function $g \in C^\infty(\mathbb{R})$ such that $g(t) = 0$ for $t \geq R$ and $g(t) = 1$ for $t \leq r$. Indeed, given such $g$ we can simply take $f(x) = g(||x||)$. To construct $g$, recall that the function

$$h(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \exp(-1/t) & \text{if } t > 0 \end{cases}$$

is smooth even at $t = 0$. The function $h(t - r) + h(R - t)$ is strictly positive everywhere, since for $t \geq r$ the first summand is positive and for $t \geq R$ the second summand is positive. Furthermore, it agrees with $h(t - r)$ for $t \geq R$. Hence the function $g \in C^\infty(\mathbb{R})$ given as

$$g(t) = 1 - \frac{h(t - r)}{h(t - r) + h(R - t)}$$

is $1$ for $t \leq r$, and $0$ for $t \leq R$.

The support $\text{supp}(f)$ of a function $f$ on $M$ is the smallest closed subset on which the function is non-zero. Equivalently, $p \in M \setminus \text{supp}(f)$ if and only if $f$ vanishes on
some open neighborhood of $p$. In the Lemma above, we can take $f$ to have support in $B_R(0)$ – simply apply the Lemma to $0 < r < R' := \frac{1}{2}(R + r)$.

**Definition A.1.** A partition of unity subordinate to an open cover $\{U_\alpha\}$ of a manifold $M$ is a collection of smooth functions $\chi_\alpha$, with $0 \leq \chi_\alpha \leq 1$, such that $\text{supp}(\chi_\alpha) \subseteq U_\alpha$, and 

$$\sum_\alpha \chi_\alpha = 1.$$

**Proposition A.3.** For any open cover of a manifold there is a partition of unity subordinate to that cover. One can take this partition of unity to be locally finite: That is, for any $p \in M$ there is an open neighborhood $U$ meeting the support of only finitely many $\chi_\alpha$’s.

**Proof.** Let $V_\beta$ be a locally finite refinement of the cover $U_\alpha$, given by coordinate charts of the kind described in Remark ??, and let $V'_\beta \subseteq V_\beta$ be as described there. Since the images of $V'_\beta \subseteq V_\beta$ are $B_\epsilon(0) \subseteq B_R(0)$, we can use the Lemma above to define a function $f_\beta \in C^\infty(M)$ with $f_\beta(p) = 1$ for $p \in V'_\beta$ and $\text{supp}(f_\beta) \subseteq V_\beta$. Since the $V'_\beta$ are already a cover, the sum $\sum_\beta f_\beta$ is everywhere.

For each index $\beta$, pick an index $\alpha$ such that $V_\beta \subseteq U_\alpha$. This defines a map $d : \beta \mapsto d(\beta)$ between the indexing sets. The functions

$$\chi_\alpha = \frac{\sum_{\beta \in d^{-1}(\alpha)} f_\beta}{\sum_{\gamma} f_\gamma}.$$  

give the desired partition of unity.