Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the previous page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.
1. (13 marks) Write one linear programming problem which satisfies all of the following:
   (i) it has two decision variables, \(x\) and \(y\)
   (ii) it is in standard form
   (iii) it has an optimal objective value
   (iv) its feasible region is unbounded
   (v) its feasible region has \(\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}\) and \(\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}\) as extreme points and these are its only extreme points.

There are infinitely many correct answers, but none is best. We give 3 solutions below. Graphs were useful in obtaining these.

**First solution:**
\[
\text{Maximize } z = -x - y \text{ s.t. } -5x + 3y \leq -15 \\
\quad x \geq 0, y \geq 0.
\]

**Second solution:**
\[
\text{Maximize } z = -y \text{ s.t. } -5x + 3y \leq -15 \\
\quad x \leq 3, y \geq 0.
\]

**Third solution:**
\[
\text{Maximize } z = -x \text{ s.t. } -5x - 3y \leq -15 \\
\quad y \leq 5, x \geq 0, y \geq 0.
\]
2.(a) (7 marks) In $\mathbb{R}^2$, let $S$ denote the solution set of the non-linear inequality, $xy \leq 1$. That is, $S = \left\{ \left[ \begin{array}{c} x \\ y \end{array} \right] \in \mathbb{R}^2 \text{ s.t. } xy \leq 1 \right\}$. Prove that $S$ is not convex.

One correct solution among infinitely many is:

$\left[ \begin{array}{c} 4 \\ 0 \end{array} \right] \in S$ and $\left[ \begin{array}{c} 0 \\ 4 \end{array} \right] \in S$ because $4 \cdot 0 = 0 \cdot 4 = 0 \leq 1$.

However, the convex combination

$$\frac{1}{2} \left[ \begin{array}{c} 4 \\ 0 \end{array} \right] + \frac{1}{2} \left[ \begin{array}{c} 0 \\ 4 \end{array} \right] = \left[ \begin{array}{c} 2 \\ 2 \end{array} \right] \notin S$$

because $2 \cdot 2 = 4 \nleq 1$.

2.(b) (7 marks) Let $S = \left\{ \left[ \begin{array}{c} x \\ y \end{array} \right] \in \mathbb{R}^2 \text{ s.t. } 9x + 5y \leq 8 \text{ and } -x + 2y \leq 7 \right\}$. Prove that

$\left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{c} -3 \\ 2 \end{array} \right]$ is not an extreme point of $S$.

One correct solution among infinitely many is given.

$\left[ \begin{array}{c} -1 \\ 3 \end{array} \right] \in S$ because $9(-1) + 5(3) = 6 \leq 8$, $-(-1) + 2(3) = 7 \leq 7$

$\left[ \begin{array}{c} -5 \\ 1 \end{array} \right] \in S$ because $9(-5) + 5(-1) = -40 \leq 8$, $-(-5) + 2(1) = 7 \leq 7$

Thus $\left[ \begin{array}{c} -3 \\ 2 \end{array} \right] = \frac{1}{2} \left[ \begin{array}{c} -1 \\ 3 \end{array} \right] + \frac{1}{2} \left[ \begin{array}{c} -5 \\ 1 \end{array} \right]$ is expressible as a convex combination of other points of $S$. Q.E.D.
3. (13 marks) Consider the following linear programming problem (in $\mathbb{R}^4$):

Minimize $z = x_1 + x_2 + x_3 + x_4$ subject to the constraints

\[
\begin{align*}
    x_1 - 2x_2 &= 4 \\
    -3x_1 + 6x_2 + x_3 + 3x_4 &= -9 \\
    2x_3 + 6x_4 &\geq 0
\end{align*}
\]

(a) (1 mark) Put the problem in canonical form.

(b) (8 marks) Find all basic solutions (feasible and infeasible) of the canonical form of the problem.

(c) (2 marks) Find all extreme points of the feasible region of the problem given above (in $\mathbb{R}^4$).

(d) (2 marks) Solve the problem given above (in $\mathbb{R}^4$). You may assume the problem has an optimal solution.

(a) With $x_5$ as a slack variable,

Maximize $Z = -x_1 - x_2 - x_3 - x_4$ s.t.

\[
\begin{align*}
    x_1 - 2x_2 &= 4 \\
    -3x_1 + 6x_2 + x_3 + 3x_4 &= -9, \quad x_i \geq 0 (i = 1, \ldots, 5)
\end{align*}
\]

(b) The coefficient matrix of the canonical problem is

\[
\begin{bmatrix}
    1 & -2 & 0 & 0 & 0 \\
    -3 & 6 & 1 & 3 & 0 \\
    0 & 0 & 2 & 6 & -1
\end{bmatrix}.
\]

The first and second columns are scalar multiples of each other, so are the third and fourth.

Thus there are only 4 basic solutions:

\[
\begin{align*}
    \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
    x_5
\end{bmatrix} &= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}.
\end{align*}
\]

(In each case, the basic variables are the non-zero variables.)

(c) Dropping the infeasible solutions and also the slack variable, the extreme points are

\[
\begin{align*}
    \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} &= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\end{align*}
\]

(d) $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is optimal (minimizes the sum of the components).