SYMPLECTIC STRATIFIED SPACES AND REDUCTION

PETER CROOKS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO

Given a Hamiltonian $G$-space $(M,\omega,\mathcal{A},\mu)$, let us consider the topological subspace $\mu^{-1}(0)$ of $M$. Since $0 \in g^*$ is a fixed point of the coadjoint representation, and since $\mu$ is $G$-equivariant, it follows that $\mathcal{A}$ restricts to a $G$-action on $\mu^{-1}(0)$. Accordingly, we may consider the quotient topological space $M_0 := \mu^{-1}(0)/G$, called the reduced space of $(M,\omega,\mathcal{A},\mu)$.

In the presence of certain additional hypotheses, $M_0$ is naturally a symplectic manifold. However, this will not hold for the general Hamiltonian $G$-space. Nevertheless, if one requires $G$ to be compact, then $M_0$ will have intriguing topological properties. In particular, there is a partition of $M_0$ into symplectic manifolds fitting together in some desirable ways. This partition realizes $M_0$ as a so-called symplectic stratified space. We will develop the notions necessary to formulate a precise definition of this object, and we will subsequently exhibit $M_0$ as a symplectic stratified space.

Definition 0.1. Let $X$ be a paracompact Hausdorff topological space, and $I$ a partially ordered set. An $I$-decomposition of $X$ is a disjoint locally finite cover, $\{S_i\}_{i \in I}$, of $X$ by locally closed subsets\(^2\), satisfying the below two properties.

(i) For each $i \in I$, the subspace $S_i$ is a topological manifold.
(ii) If $(i,j) \in I \times I$, then $S_i \cap S_j \neq \emptyset \iff S_i \subseteq S_j \iff i \leq j.\(^3\)

A decomposed space is a paracompact Hausdorff topological space $X$, together with a distinguished $I$-decomposition $\{S_i\}_{i \in I}$ of $X$ for some partially ordered set $I$. We shall call the partially ordered set $I$ the index set of the decomposed space, and the subspaces $S_i$ the pieces of the space.

Example 0.1. Let $X$ be a topological space. Recall that the cone over $X$, $\text{CX}$, is defined to be the quotient of $X \times [0,\infty)$ obtained by identifying the points in $X \times \{0\}$. If $(X,\{S_i\}_{i \in I})$ is a decomposed space, then there is a canonical realization of $\text{CX}$ as a decomposed space. Precisely, one defines the set $J = I \cup \{0\}$, and augments it with the partial order coinciding with that on $I \subseteq J$, such that $0 \leq i$ for all $i \in I$. Now, for each $i \in I$, define $\tilde{S}_i$ to be the image of $S_i \times (0,\infty)$ under the quotient map $X \times [0,\infty) \to \text{CX}$. Also, let $\tilde{S}_0$ be the image of $X \times \{0\}$ under the quotient map. We note that $\{\tilde{S}_j\}_{j \in J}$ is a $J$-decomposition of $\text{CX}$, as desired.

Definition 0.2. Let $X$ be a decomposed space, and $S \subseteq X$ a piece. We define an $S$-chain in $X$ of length $n \geq 0$ to be a sequence, $S = A_0, A_1, \ldots, A_n$, of $n+1$ pieces, with the property that if $i, j \in \{0, \ldots, n\}$ and $j = i + 1$, then $A_i \neq A_j$ and $A_i \subseteq \overline{A_j}$. The depth of $S$, $\text{depth}_X(S)$, is defined to be

\[
\text{depth}_X(S) = \sup\{n \geq 0 : \exists \text{ an } S\text{-chain of length } n\}.
\]

---

\(^1\) Indeed, if $G$ is compact and acts freely on $\mu^{-1}(0)$, then $\mu^{-1}(0)$ is an embedded submanifold of $M$, there exists a unique smooth manifold structure on $M_0$ for which the quotient map $\pi : \mu^{-1}(0) \to M_0$ is a submersion, and there is a unique symplectic form $\omega_0$ on the smooth manifold $M_0$ for which $\pi^*(\omega_0)$ is the restriction of $\omega$ to $\mu^{-1}(0)$. This is a statement of the Marsden-Weinstein-Meyer Theorem.

\(^2\) A subset of a topological space is called locally closed if it is open with respect to the subspace topology of its closure.

\(^3\) One calls this the Frontier Condition.
Definition 0.3. Let $X$ be a decomposed space with non-empty index set $I$ and pieces $\{S_i\}_{i \in I}$. The depth of $X$, $\text{depth}(X)$, is defined by

$$\text{depth}(X) = \sup_{i \in I} \text{depth}_X(S_i).$$

Remark 0.1. In the interest of our being able to define the depth of an arbitrary decomposed space, we shall require that each of our decomposed spaces come equipped with a non-empty index set.

Definition 0.4. A 0-stratified space is a decomposed space $X$ of depth 0. An $n$-stratified space, $n \geq 1$, is a decomposed space $(X, \{S_i\}_{i \in I})$ of depth $n$, with the property that for each piece $S$ of $X$ and point $x \in S$, there exist an open neighbourhood $U(x)$ of $x$ in $X$, an open coordinate ball $B(x)$ of $x$ in $S$, an $m$-stratified stratified space $(L, \{P_j\}_{j \in J})$ with $m < n$, and a homeomorphism $\varphi_x : B(x) \times CL \to U(x)$, such that for each piece of $B(x) \times CL$, $\varphi_x$ restricts to a homeomorphism of that piece with a piece of $U(x)$.

We shall refer to the pieces of a stratified space as strata.

Definition 0.5. A smooth stratified space consists of a stratified space $X$, together with the below data.

(i) a smooth manifold structure for each stratum of $X$

(ii) a distinguished subalgebra, $C^\infty(X)$, of the $\mathbb{R}$-algebra $C^0(X)$ of continuous maps $X \to \mathbb{R}$, with the property that $f|_S \in C^\infty(S)$ for all strata $S$ of $X$ and for all $f \in C^\infty(X)$

Definition 0.6. A symplectic stratified space consists of a smooth stratified space $X$, augmented with the below additional data.

(i) a symplectic form, $\omega_S \in \Omega^2(S)$, for each stratum $S$ of $X$

(ii) a Poisson algebraic structure\(^4\), $\{\cdot, \cdot\} : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ on $C^\infty(X)$, for which the restriction maps to strata, $i_S^* : C^\infty(X) \to C^\infty(S)$, are Poisson algebra morphisms\(^5\)

Let $G$ be a group and $M$ a set with a left $G$-action. We wish to associate a canonical partially ordered set to this action. To this end, denote by $G^S$ the collection of those subgroups of $G$ with the property of being conjugate in $G$ to the stabilizer subgroup of a point in $M$. More succinctly,

$$G^S := \{H \leq G : \exists p \in M, g \in G \text{ such that } gHg^{-1} = \text{Stab}_G(p)\}.$$ 

Identifying conjugate subgroups of $G^S$, we obtain an equivalence relation. Let $I$ denote the resulting quotient space. We define a partial order, $\leq$, on $I$ by $[H] \leq [K]$ if and only if $K$ is contained in a conjugate of $H$ in $G$. Well-definedness follows from the observation that $K$ is contained in a conjugate of $H$ if and only if for every conjugate $H'$ of $H$ and $K'$ of $K$, $K'$ is contained in a conjugate of $H'$.

For each $\alpha \in I$, consider the set $M_\alpha := \{p \in M : [\text{Stab}_G(p)] = \alpha\}$. Let us specialize to the case in which $G$ is a compact Lie group with Lie algebra $\mathfrak{g}$, and $(M, \omega, \mathbf{A}, \mu)$ is a Hamiltonian $G$-space. For future reference, we shall let $Z := \mu^{-1}(0)$, the zero-level set of the moment map. Consider the quotient map $\pi : Z \to M_0$, and for each $\alpha \in I$, set $(M_0)_\alpha := \pi(M_\alpha \cap Z)$. We observe that if $\alpha, \beta \in I$ and $(M_0)_\alpha \cap (M_0)_\beta \neq \emptyset$, then we may choose $p \in M_\alpha \cap Z$ and $q \in M_\beta \cap Z$, such that $\pi(p) = \pi(q)$. By the definition of our quotient space $M_0$, it follows that $p$ and $q$ lie in the same $G$-orbit, and hence $\text{Stab}_G(p)$ and $\text{Stab}_G(q)$ are conjugate in $G$. Therefore, $[\text{Stab}_G(p)] = [\text{Stab}_G(q)]$ in $I$. However, $p \in M_\alpha$ and $q \in M_\beta$, implying that $\alpha = [\text{Stab}_G(p)]$ and $\beta = [\text{Stab}_G(q)]$. It follows that $\alpha = \beta$, and we conclude that the sets $\{(M_0)_i : i \in I\}$ are disjoint. Furthermore, the sets $M_i$ cover $M$, meaning that the sets $M_i \cap Z$ cover $Z$, and hence that the sets $(M_0)_i$ cover $M_0$ (as $\pi$ is surjective).

In light of our determinations, it perhaps seems sensible to regard the $(M_0)_\alpha$’s as candidates for strata of the reduced space $M_0$. However, there is an example of a reduced space in which one of these

\(^4\)The decomposed space structures of $B(x) \times CL$ and $U(x)$ are canonically induced by those of $CL$ and $X$, respectively. Specifically, the pieces of $B(x) \times CL$ are $\{B(x) \times B_j\}_{j \in J(x)}$, while those of $U(x)$ are $\{U(x) \cap S_i\}_{i \in I}$.

\(^5\)A Poisson algebra over a field $K$ is an associative $K$-algebra $A$, together with a Lie bracket on $A$ that is simultaneously a derivation of $A$.

\(^6\)We view $C^\infty(S)$ as the Poisson algebra canonically induced by the symplectic form $\omega_S$. 

2
candidate strata has connected components of different dimensions (meaning that this stratum is not a topological manifold). Fortunately, some semblance of a resolution is obtained via partitioning the candidate strata into connected components.

**Theorem 0.1.** The reduced space $M_0$ is a disjoint union of the subspaces $\{(M_0)_\alpha : \alpha \in I\}$. This decomposition has the below properties.

(i) If $\alpha \in I$, then each connected component of $(M_0)_\alpha$ is a topological manifold.

(ii) If $(\alpha, \beta) \in I \times I$, then $\alpha \leq \beta \Rightarrow (M_0)_\alpha \cap (M_0)_\beta \neq \emptyset \Leftrightarrow (M_0)_\alpha \cap (M_0)_\beta \neq \emptyset$ and every connected component of $(M_0)_\alpha$ intersecting $(M_0)_\beta$ non-trivially belongs to $(M_0)_\beta$.

(iii) There is a canonical realization of $M_0$ as a symplectic stratified space with strata the connected components of the $(M_0)_\alpha$'s.\(^7\)

**Claim 0.1.** If $\alpha \in I$ and $p \in (M_0)_\alpha \cap Z$, then there is an open subset $U \subseteq (M_0)_\alpha$ containing $[p]$, and a realization of the subspace $U$ as a symplectic manifold.

Given $\alpha \in I$ and $p \in (M_0)_\alpha \cap Z$, let $O_p$ denote the $G$-orbit of $p$ in $M$. Since $G$ is compact, $O_p$ is an embedded submanifold of $M$. More intriguingly, perhaps, this embedding is isotropic (the proof of which was given in the presentation).

**Lemma 0.1.** The embedding $i : O_p \hookrightarrow M$ is isotropic.

**Theorem 0.2.** (Weinstein’s Equivariant Isotropic Embedding Theorem) Let $K$ be a compact Lie group, $B$ a smooth $K$-manifold, and $(E, \omega)$, $(E', \omega')$ symplectic manifolds, each augmented with a $K$-action by symplectic automorphisms. Suppose that $i : B \hookrightarrow E$ and $i' : B \hookrightarrow E'$ are $K$-equivariant isotropic embeddings with isomorphic symplectic normal bundles.\(^8\) Then, there exist $K$-invariant open neighbourhoods, $U$ and $U'$, of $i(B)$ in $E$ and $i'(B)$ in $E'$, respectively, and a $K$-equivariant symplectomorphism, $\varphi : U \rightarrow U'$, such that $\varphi \circ i = i'$ as maps $B \rightarrow E'$.

With the Equivariant Isotropic Embedding Theorem in mind, we observe that the inclusion $O_p \hookrightarrow M$ is a $G$-equivariant isotropic embedding of $O_p$ into a symplectic manifold. Seeking to apply our theorem, we will $G$-equivariantly and isotropically embed $O_p$ into another symplectic $G$-manifold, such that the associated symplectic normal bundle is isomorphic to that of the embedding $O_p \hookrightarrow M$.

To this end, consider the fibre $V := (N^\omega O_p)_p$ of the symplectic normal bundle of $O_p \hookrightarrow M$. It is easily verified that $\omega(p)$ descends to a symplectic form on $V$. Setting $H := Stab_G(p)$, we note that $H$ acts on $V$ by symplectic vector space automorphisms. Now, let $\mathfrak{h} = \text{Lie}(H) \subseteq \mathfrak{g}$, noting that $H$ is a closed subgroup (hence an embedded submanifold) of $G$. Note that $\mathfrak{h}$ is an invariant subspace of the restricted adjoint representation $H \rightarrow Aut(\mathfrak{g})$, allowing for us to induce an $H$-representation on $\mathfrak{g}/\mathfrak{h}$. Of course, one then has the canonical dual representation on $(\mathfrak{g}/\mathfrak{h})^\ast$. Furthermore, we may consider the direct sum $(\mathfrak{g}/\mathfrak{h})^\ast \oplus V$ of linear $H$-representations.

Now, consider the principal $H$-bundle $G \rightarrow O_p$, $g \mapsto g \cdot p$, and form the so-called associated bundle $Y := G \times_H ((\mathfrak{g}/\mathfrak{h})^\ast \oplus V)$. Recall that $Y$ is the product manifold $G \times ((\mathfrak{g}/\mathfrak{h})^\ast \oplus V)$, modulo the free left $H$-action $h \cdot (g, v) = (gh^{-1}, h \cdot v)$. It is natural, then, to consider the map $\pi : Y \rightarrow O_p$ given by $[(g, v)] \mapsto g \cdot p$. This constitutes a vector bundle with total space $Y$ and base space $O_p$. Accordingly,

\(^7\)Since we do not claim to have exhibited $M_0$ as a decomposed space in the sense of Definition 1.2, we must specify precisely what is meant by item (iii). To this end, we mean that each of our advertised strata has a canonical symplectic manifold structure, and that $M_0$ has a canonical $C^\infty(M_0)$-subalgebra, $C^\infty(M_0)$, with a Poisson bracket for which the restriction maps to strata $S$ define Poisson algebra morphisms $C^\infty(M_0) \rightarrow C^\infty(S)$.

\(^8\)Recall that the symplectic perpendicular of the embedding $i : B \hookrightarrow E$, $T^\omega B$, is the subbundle of the restricted tangent bundle $TE|_B$ with fibres $(T^\omega B)_p = \{v \in T_p E : \omega(p)(v, w) = 0 \forall w \in T_p B\}$, $p \in B$. The symplectic normal bundle of the isotropic embedding, $N^\omega(B)$, is then defined to be the quotient bundle $N^\omega(B) := T^\omega B/\mathcal{B}$, noting that our embedding induces an inclusion $TB \subseteq T^\omega(B) \subseteq TE|_B$. 

3
we consider the zero-section embedding $s : \mathcal{O}_p \to Y$, $g \cdot p \mapsto [(g, 0)]$. If one endows $Y$ with the left $G$-action $g \cdot [(g', v)] = [(gg', v)]$, then $s$ becomes a $G$-equivariant embedding. It therefore remains to exhibit $Y$ as a symplectic manifold, such that $G$ acts on $Y$ by symplectic automorphisms, and such that $s$ is an isotropic embedding whose symplectic normal bundle is isomorphic to that of $\mathcal{O}_p \to M$.

Consider the trivialization $\Psi : G \times \mathfrak{g}^* \to T^* G$ of the cotangent bundle of $G$ defined by $\Psi(g, \theta) = (g, \theta \circ dL_{g^{-1}}(g))$. One then considers the $G$-action on $G \times \mathfrak{g}^*$ given by $g \cdot (g', \theta) = (gg^{-1}, Ad^*(g)(\theta))$, where $Ad^* : G \to \text{Aut}(\mathfrak{g}^*)$ is the coadjoint representation of $G$. Deploying our trivialization, we obtain a Hamiltonian $G$-action on $T^* G$ (where we are regarding $T^* G$ as augmented with its canonical symplectic form). This restricts to a Hamiltonian $H$-action, as an associated moment map is obtained by composing the previous moment map with the projection $\mathfrak{g}^* \to \mathfrak{h}^*$.

Now, recall that $H$ acts on $V$ by symplectic vector space automorphisms. Indeed, this action is actually Hamiltonian. Accordingly, it will be advantageous to consider the Hamiltonian $H$-space $T^* G \times V$. To see that this $H$-action is free, suppose $h \in H$ fixes $((g, \theta), v) \in (G \times \mathfrak{g}^*) \times V \cong T^* G \times V$. By definition, $((gh^{-1}, Ad^*(h)(\theta)), h \cdot v) = ((g, \theta), v)$. In particular, $g = gh^{-1}$, meaning that $h = e$. Note also that $H$ is a compact Lie group by virtue of being a closed subspace of the compact Lie group $G$. The Marsden-Weinstein-Meyer Theorem therefore gives a canonical symplectic manifold structure on the reduced space $\Phi^{-1}(0)/H$, where $\Phi : T^* G \times V \to \mathfrak{h}^*$ is the moment map.

Next, one constructs an $H$-equivariant diffeomorphism, $G \times ((\mathfrak{g}/\mathfrak{h})^* \oplus V) \to \Phi^{-1}(0)$, and obtains an induced diffeomorphism $Y = G \times ((\mathfrak{g}/\mathfrak{h})^* \oplus V)/H \to \Phi^{-1}(0)/H$. Hence, we endow $Y$ with the symplectic manifold structure for which this diffeomorphism is a symplectomorphism. We leave it to the interested reader to verify that $G$ acts on $Y$ by symplectomorphisms, and that $s : \mathcal{O}_p \to Y$ is an isotropic embedding with symplectic normal bundle isomorphic to that of $\mathcal{O}_p \to M$.

By Theorem 1.1, we may choose $G$-invariant open submanifolds $U$ and $U'$ of $\mathcal{O}_p$ in $M$ and of the zero-section in $Y$, respectively, and a $G$-equivariant symplectomorphism $\varphi : U \to U'$ respecting the embeddings $\mathcal{O}_p \to M$ and $\mathcal{O}_p \to Y$. The $G$-action on $Y$ is incidentally Hamiltonian, with a moment map $J : Y \to \mathfrak{g}^*$ explicitly constructed in [3]. Hence, $J|_{U'} \circ \varphi : U \to \mathfrak{g}^*$ is a moment map of the Hamiltonian $G$-action on $U$, meaning that $\mu|_U = J|_{U'} \circ \varphi + f$ for some constant map $f : U \to \mathfrak{g}^*$. Since $\mu(p) = 0$, it follows that $f = -J(\varphi(p))$. Because $\varphi$ respects the embeddings $\mathcal{O}_p \to M$ and $\mathcal{O}_p \to Y$, $\varphi(p)$ belongs to the zero-section of the vector bundle $Y \to \mathcal{O}_p$. However, the moment map $J$ vanishes on the zero-section, meaning that $f = -J(\varphi(p)) = 0$. It follows that $J|_{U'} \circ \varphi = \mu|_U$. Therefore, $\varphi$ is an isomorphism of the Hamiltonian $G$-spaces $(U, \mu|_U)$ and $(U', J|_{U'})$. In particular, for a given $\alpha \in I$, $\varphi$ must therefore induce an identification of the quotients $(U_\alpha \cap \mu^{-1}(0))/G = (M_\alpha \cap U \cap \mu^{-1}(0))/G$ and $(U'_\alpha \cap J^{-1}(0))/G = (Y_\alpha \cap U' \cap J^{-1}(0))/G$. We will realize $(Y_\alpha \cap U' \cap J^{-1}(0))/G$ as a symplectic manifold and our identification will then induce a symplectic manifold structure on $(M_\alpha \cap U \cap \mu^{-1}(0))/G$. Since the quotient projection $\pi_0 : M_\alpha \cap \mu^{-1}(0) \to (M_\alpha \cap \mu^{-1}(0))/G = (M_\alpha)_0$ is an open map, $(M_\alpha \cap U \cap \mu^{-1}(0))/G$ is an open subset of $(M_\alpha)_0$, and we will therefore have realized an open neighbourhood of an arbitrary point of $(M_\alpha)_0$ as a symplectic manifold.

Since the quotient projection $Y_\alpha \cap J^{-1}(0) \to (Y_\alpha \cap J^{-1}(0))/G$ is also an open map, it follows that $(Y_\alpha \cap U' \cap J^{-1}(0))/G$ is an open subset of $(Y_\alpha \cap J^{-1}(0))/G$. Accordingly, it will suffice to exhibit $(Y_\alpha \cap J^{-1}(0))/G$ as a symplectic manifold, as one will then obtain an induced symplectic structure on the open submanifold $(Y_\alpha \cap U' \cap J^{-1}(0))/G$.

Now, consider the linear subspace $V_H := \{v \in V : h \cdot v = v \ \forall h \in H\}$ of $V$. It is easily established that the restriction of the symplectic form on $V$ to $V_H$ yields a symplectic form on $V_H$. This realizes $V_H$ as a symplectic manifold. Furthermore, the authors in [3] use properties of the moment map $J$ to identify the quotient $(Y_\alpha \cap J^{-1}(0))/G$ with $V_H$, and in so doing, they endow this quotient with the structure of a symplectic manifold (as desired). We have thus outlined the proof of our claim.

Let us briefly address the symplectic structure on $M_0$. To this end, let $\pi : Z \to M_0$ be the quotient map, and define $f \in C^\infty(M_0)$ to be an element of $C^\infty(M_0)$ if $f \circ \pi = F|_Z$ for some $F \in C^\infty(M)^G$. The Poisson bracket, $\{f, g\}_{M_0}$, of $f, g \in C^\infty(M_0)$ is given by $\{f, g\}_{M_0}(p) = \{f|_S, g|_S\}_S(p)$, where $p \in M_0$,
$S$ is the stratum of $M_0$ containing $p$, and $\{,\}_S : C^\infty(S) \times C^\infty(S) \to C^\infty(S)$ is the Poisson bracket on $C^\infty(S)$. 
REFERENCES