1. The purpose of this exercise is to establish the properties of the sign of a permutation. Recall that \( S_n \) denotes the set of all permutations of \( \{1, \ldots, n\} \). If \( \sigma, \tau \) are two permutations, then \( \sigma \tau \) denotes their composition, so \( (\sigma \tau)(i) = \sigma(\tau(i)) \). We will write \( id \) for the identity permutation. In this exercise, do not use the determinant of matrices.

(a) A simple transposition is a permutation \( \sigma \in S_n \) which exchanges a pair of neighbouring elements. More precisely, there exists \( i \) with

\[
\begin{align*}
\sigma(i + 1) &= i \\
\sigma(i) &= i + 1 \\
\sigma(j) &= j \quad \text{otherwise.}
\end{align*}
\]

Prove that if \( \sigma_1, \ldots, \sigma_k \) are all simple transpositions and \( \sigma_1 \sigma_2 \cdots \sigma_k = id \), then \( k \) is even. Hint: one way to do this is to consider the length function \( \ell(\sigma) = |\{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}| \).

(b) Use (a) to prove that there exists a unique function \( sign : S_n \to \{1, -1\} \) such that \( sign(\sigma \tau) = sign(\sigma) sign(\tau) \) for any two permutations and \( sign(\sigma) = -1 \) whenever \( \sigma \) is a simple transposition.

(c) Prove that \( sign(\sigma) = -1 \) whenever \( \sigma \) is any transposition.

2. Let \( V, W \) be two vector spaces. Let \( B(V, W) \) denote the set of all bilinear pairings \( B : V \times W \to \mathbb{F} \).

(a) Construct a natural linear map \( V^* \otimes W^* \to B(V, W) \).
(b) Prove that this map is an isomorphism when $V, W$ are finite-dimensional.

(c) Suppose that $V, W$ are finite-dimensional. Construct a natural isomorphism $V \otimes W \rightarrow B(V^*, W^*)$. (This is the definition of $V \otimes W$ from *Linear Algebra Done Wrong*.)

3. Let $F = F_2 = \{0, 1\}$ the field with two elements. In this field $1 + 1 = 0$.

Let $V = F^2$. Consider the linear map $\tau : V \otimes V \rightarrow V \otimes V$, defined by $\tau(v \otimes w) = w \otimes v$. Find a basis $\beta$ for $V \otimes V$ such that $[\tau]_\beta$ is a Jordan form matrix.

4. Let $V$ be a vector space with basis $\{x_1, \ldots, x_n\}$. Then every element of $V \otimes V$ can be written uniquely as $y = \sum_{1 \leq i, j \leq n} c_{ij} x_i \otimes x_j$. So every element of $V \otimes V$ can be represented by a matrix $C = (c_{ij})$. Prove that $y \in Sym^2V$ if and only if $C$ is symmetric and that $y \in \Lambda^2V$ if and only if $C$ is skew-symmetric.