1. Let \( \{V_i\}_{i \in I} \) be a collection (possibly infinite) of vector spaces. There are two ways to take the “direct sum” of all these vector spaces. First we have the direct sum

\[
\bigoplus_{i \in I} V_i := \{(v_i)_{i \in I} : v_i \in V_i \text{ and } v_i \text{ is non-zero for only finitely-many } i\}
\]

and we have the direct product

\[
\prod_{i \in I} V_i := \{(v_i)_{i \in I} : v_i \in V_i\}
\]

In each case, they are vector spaces, with addition and scalar multiplication defined in the obvious way.

In each case we have inclusion map \( \phi_i : V_i \to \bigoplus_{i \in I} V_i \) and \( \phi_i : V_i \to \prod_{i \in I} V_i \) and projection maps \( \psi_i : \bigoplus_{i \in I} V_i \to V_i \) and \( \psi_i : \prod_{i \in I} V_i \to V_i \).

For each of the two following statements, fill in the blank with either the direct sum or the direct product and then prove the statement.

(a) Let \( X \) be a vector space and let \( T_i : V_i \to X \) be linear maps for all \( i \in I \). There exists a unique linear map \( T : \quad \to X \) such that \( T_i = T \circ \phi_i \) for all \( i \).

(b) Let \( X \) be a vector space and let \( U_i : X \to V_i \) be linear maps for all \( i \in I \). There exists a unique linear map \( U : X \to \quad \) such that \( U_i = \psi_i \circ U \) for all \( i \).
2. Let $I$ be any set and let
$$\mathbb{F}[I] = \{(a_i)_{i \in I} : a_i \in \mathbb{F} \text{ is non-zero for only finitely-many } i\}$$

Let $e_i \in \mathbb{F}[I]$ be the “tuple” which is 1 in the $i$th slot and 0 elsewhere.

Let $X$ be a vector space and for each $i \in I$, let $x_i \in X$. Prove that there exists a unique linear map $T : \mathbb{F}[I] \to X$ such that $T(e_i) = x_i$ for all $i \in I$.

3. Given an example of an element of $\mathbb{F}^2 \otimes \mathbb{F}^2$ which cannot be written as $v \otimes w$.

4. Let $V$ and $W$ be vector spaces. If $\alpha \in V^*$ and $w \in W$, define $T_{\alpha,w} : V \to W$ by $T_{\alpha,w}(v) = \alpha(v)w$.

(a) Prove that for any $\alpha, w$, $T_{\alpha,w}$ is a linear map.

(b) Define a linear map $\psi : V^* \otimes W \to L(V, W)$ by $\psi(\alpha \otimes w) = T_{\alpha,w}$. Prove that $\psi$ is well-defined and that it is an isomorphism of vector spaces when $V, W$ are finite-dimensional.

(c) Let $T \in L(V, W)$. Prove that $T = \psi(\alpha \otimes w)$ for some $\alpha \in V^*$, $w \in W$ if and only if $\text{rank}(T) \leq 1$.

5. (a) Let $A$ and $B$ be upper-triangular square matrices. Prove that $A \otimes B$ is also upper triangular.

(b) Let $T : V \to V$ and $U : W \to W$ be linear operators. We have the linear operator $T \otimes U : V \otimes W \to V \otimes W$. If $\lambda$ is an eigenvalue of $T$ and $\mu$ is an eigenvalue of $U$, prove that $\lambda\mu$ is an eigenvalue of $T \otimes U$.

(c) Assume $\mathbb{F} = \mathbb{C}$. Use (a) to prove that every eigenvalue of $T \otimes U$ can be written as $\lambda\mu$ where $\lambda$ is an eigenvalue of $T$ and $\mu$ is an eigenvalue of $U$. 

2