1. Let $T$ be a linear operator on a complex vector space $V$. Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $T$.

   (a) Let $v \in V$. Prove that there exist unique vectors $v_1, \ldots, v_k$ such that $v = v_1 + \cdots + v_k$ and $v_i \in K_{\lambda_i}$ for all $i$.

   (b) Define a linear operator $D : V \to V$ by $D(v) = \lambda_1 v_1 + \cdots + \lambda_k v_k$ and let $N = T - D$. Prove that $D$ is diagonalizable, $N$ is nilpotent, and $DN = ND$.

   (c) Pick a basis $\beta$ for $V$ for which $[T]_\beta$ is a Jordan form matrix. Describe $[D]_\beta$ and $[N]_\beta$.

2. Let $A$ be an $n \times n$ matrix with real entries. Prove that the minimal polynomial of $A$ when considered as a real matrix is the same as the minimal polynomial of $A$ when considered as a complex matrix.

3. (a) Using the property $\det(AB) = \det(A)\det(B)$ prove that the determinant of a matrix is unchanged if we add a multiple of one row to another row.

   (b) Find the determinant of the following matrix

   \[
   \begin{bmatrix}
   1 & 1 & 1 \\
   1 & 2 & 1 \\
   3 & 3 & 7
   \end{bmatrix}
   \] (1)

4. Let $P_n$ denote the vector space of polynomials (with complex coefficients) of degree at most $n - 1$. Let $c_1, \ldots, c_n$ denote $n$ complex numbers. Define a linear map $T : P_n \to \mathbb{C}^n$ by $T(f) = (f(c_1), \ldots, f(c_n))$. 

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(a) Let $\alpha = \{1, x, \ldots, x^{n-1}\}$ be the usual basis for $P_n$ and let $\beta$ be the standard basis for $\mathbb{C}^n$. Prove that

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & c_1 & \cdots & c_1^{n-1} \\ 1 & c_2 & \cdots & c_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & \cdots & c_n^{n-1} \end{bmatrix}$$

(b) Prove that the determinant of the above matrix is

$$\prod_{1 \leq i < j \leq n} (c_j - c_i).$$

[Hint: first use column operations to make the first row $[1 \ 0 \ \ldots \ 0]$ and then evaluate the determinant by expanding along the first row.]

(c) Prove that $T$ is invertible if and only if $c_1, \ldots, c_n$ are distinct.