Multilinear forms

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Assume that all fields are characteristic 0 (i.e. $1 + \cdots + 1 \neq 0$), for example $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Assume also that all vector spaces are finite dimensional.

1 Dual spaces

If $V$ is a vector space, then $V^* = L(V, \mathbb{F})$ is defined to be the space of linear maps from $V$ to $\mathbb{F}$.

If $v_1, \ldots, v_n$ is a basis for $V$, then we define $\alpha_i \in V^*$ for $i = 1, \ldots, n$, by setting

$$\alpha_i(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

**Proposition 1.1.** $\alpha_1, \ldots, \alpha_n$ forms a basis for $V^*$ (called the dual basis).

In particular, this shows that $V$ and $V^*$ are vector spaces of the same dimension. However, there is no natural way to choose an isomorphism between them, unless we pick some additional structure on $V$ (such as a basis or a non-degenerate bilinear form).

On the other hand, we can construct an isomorphism $\psi$ from $V$ to $(V^*)^*$ as follows. If $v \in V$, then we define $\psi(v)$ to be the element of $V^*$ given by

$$(\psi(v))(\alpha) = \alpha(v)$$

for all $\alpha \in V^*$. In other words, given a guy in $V$, we tell him to eat elements in $V^*$ by allowing himself to be eaten.

**Proposition 1.2.** $\psi$ is an isomorphism.

**Proof.** Since $V$ and $(V^*)^*$ have the same dimension, it is enough to show that $\psi$ is injective.

Suppose that $v \in V$, $v \neq 0$, and $\psi(v) = 0$. We wish to derive a contradiction.

Since $v \neq 0$, we can extend $v$ to a basis $v_1 = v, v_2, \ldots, v_n$ for $V$. Then let $\alpha_1$ defined as above. Then $\alpha_1(v) = 1 \neq 0$ and so we have a contradiction. Thus $\psi$ is injective as desired.

From this proposition, we derive the following useful result.
Corollary 1.3. Let $\alpha_1, \ldots, \alpha_n$ be a basis for $V^*$. Then there exists a basis $v_1, \ldots, v_n$ for $V$ such that

$$\alpha_i(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

for all $i, j$.

Proof. Let $w_1, \ldots, w_n$ be the dual basis to $\alpha_1, \ldots, \alpha_n$ in $(V^*)^\ast$. Since $\psi$ is invertible, $\psi^{-1}$ exists. Define $v_i = \psi^{-1}(w_i)$. Since $w_1, \ldots, w_n$ is a basis, so is $v_1, \ldots, v_n$. Checking through the definitions shows that $v_1, \ldots, v_n$ have the desired properties. \qed

2 Bilinear forms

Let $V$ be a vector space. We denote the set of all bilinear forms on $V$ by $(V^*)^\otimes 2$. We have already seen that this set is a vector space.

Similarly, we have the subspaces $\text{Sym}^2 V^*$ and $\Lambda^2 V^*$ of symmetric and skew-symmetric bilinear forms.

Proposition 2.1. $(V^*)^\otimes 2 = \text{Sym}^2 V^* \oplus \Lambda^2 V^*$

Proof. Clearly, $\text{Sym}^2 V^* \cap \Lambda^2 V^* = 0$, so it suffices to show that any bilinear form is the sum of a symmetric and skew-symmetric bilinear form. Let $H$ be a bilinear form. Let $\hat{H}$ be the bilinear form defined by

$$\hat{H}(v_1, v_2) = H(v_2, v_1)$$

Then $(H + \hat{H})/2$ is symmetric and $(H - \hat{H})/2$ is skew-symmetric. Hence $H = (H + \hat{H})/2 + (H - \hat{H})/2$ is the sum of a symmetric and skew-symmetric form. \qed

If $\alpha, \beta \in V^*$, then we can define a bilinear form $\alpha \otimes \beta$ as follows.

$$(\alpha \otimes \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2)$$

for $v_1, v_2 \in V$.

We can also define a symmetric bilinear form $\alpha \cdot \beta$ by

$$(\alpha \cdot \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) + \alpha(v_2)\beta(v_1)$$

and a skew-symmetric bilinear form $\alpha \wedge \beta$ by

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1)$$

These operations are linear in each variable. In other words

$$\alpha \otimes (\beta + \gamma) = \alpha \otimes \beta + \alpha \otimes \gamma$$

and similar for the other operations.
Example 2.2. Take $V = \mathbb{R}^2$. Let $\alpha_1, \alpha_2$ be the standard dual basis for $V^*$, so that

$$\alpha_1(x_1, x_2) = x_1, \quad \alpha_2(x_1, x_2) = x_2$$

Then $\alpha_1 \otimes \alpha_2$ is given by

$$(\alpha_1 \otimes \alpha_2)((x_1, x_2), (y_1, y_2)) = x_1 y_2$$

Similarly $\alpha_1 \wedge \alpha_2$ is the standard symplectic form on $\mathbb{R}^2$, given by

$$(\alpha_1 \wedge \alpha_2)((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1$$

$\alpha_1 \cdot \alpha_2$ is the symmetric bilinear form of signature $(1, 1)$ on $\mathbb{R}^2$ given by

$$(\alpha_1 \cdot \alpha_2)((x_1, x_2), (y_1, y_2)) = x_1 y_2 + x_2 y_1$$

The standard positive definite bilinear form on $\mathbb{R}^2$ (the dot product) is given by $\alpha_1 \cdot \alpha_1 + \alpha_2 \cdot \alpha_2$.

3 Multilinear forms

Let $V$ be a vector space.

We can consider $k$-forms on $V$, which are maps

$$H : V \times \cdots \times V \to \mathbb{F}$$

which are linear in each argument. In other words

$$H(av_1, \ldots, v_k) = aH(v_1, \ldots, v_k)$$

$$H(v + w, v_2, \ldots, v_k) = H(v, v_2, \ldots, v_k) + H(w, v_2, \ldots, v_k)$$

for $a \in \mathbb{F}$ and $v, w, v_1, \ldots, v_k \in V$, and similarly in all other arguments.

$H$ is called symmetric if for each $i$, and all $v_1, \ldots, v_k$,

$$H(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \ldots, v_n) = H(v_1, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots, v_n)$$

$H$ is called skew-symmetric (or alternating) if for each $i$, and all $v_1, \ldots, v_k$,

$$H(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \ldots, v_n) = -H(v_1, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots, v_n)$$

The vector space of all $k$-forms is denoted $(V^*)^\otimes k$ and the subspaces of symmetric and skew-symmetric forms are denote $\text{Sym}^k V^*$ and $\Lambda^k V^*$.

3.1 Permutations

Let $S_k$ denote the set of bijections from $\{1, \ldots, k\}$ to itself (called a permutation). $S_k$ is also called the symmetric group. It has $k!$ elements. The permutations occurring in the definition of symmetric and skew-symmetric forms are
called simple transpositions (they just swap \(i\) and \(i + 1\)). Every permutation can be written as a composition of simple transpositions.

From this it immediately follows that if \(H\) is symmetric and if \(\sigma \in S_k\), then
\[
H(v_1, \ldots, v_k) = H(v_{\sigma(1)}, \ldots, v_{\sigma(n)})
\]

There is a function \(\varepsilon : S_k \to \{1, -1\}\) called the sign of a permutation, which is defined by the conditions that \(\varepsilon(\sigma) = -1\) if \(\sigma\) is a simple transposition and
\[
\varepsilon(\sigma_1 \sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2)
\]
for all \(\sigma_1, \sigma_2 \in S_k\).

The sign of a permutation gives us the behaviour of skew-symmetric \(k\)-forms under permuting the arguments. If \(H\) is skew-symmetric and if \(\sigma \in S_k\), then
\[
H(v_1, \ldots, v_k) = \varepsilon(\sigma)H(v_{\sigma(1)}, \ldots, v_{\sigma(n)})
\]

3.2 Iterated tensors, dots, and wedges

If \(H\) is a \(k-1\)-form and \(\alpha \in V^*\), then we define \(H \otimes \alpha\) to be the \(k\)-form defined by
\[
(H \otimes \alpha)(v_1, \ldots, v_k) = H(v_1, \ldots, v_{k-1})\alpha(v_k)
\]

Similarly, if \(H\) is a symmetric \(k-1\)-form and \(\alpha \in V^*\), then we define \(H \cdot \alpha\) to be the \(k\)-form defined by
\[
(H \cdot \alpha)(v_1, \ldots, v_k) = H(v_1, \ldots, v_{k-1})\alpha(v_k) + \cdots + H(v_2, \ldots, v_k)\alpha(v_1)
\]
It is easy to see that \(H \cdot \alpha\) is a symmetric \(k\)-form.

Similarly, if \(H\) is a skew-symmetric \(k-1\)-form and \(\alpha \in V^*\), then we define \(H \wedge \alpha\) to be the \(k\)-form defined by
\[
(H \wedge \alpha)(v_1, \ldots, v_k) = H(v_1, \ldots, v_{k-1})\alpha(v_k) - \cdots \pm H(v_2, \ldots, v_k)\alpha(v_1)
\]
It is easy to see that \(H \wedge \alpha\) is a skew-symmetric \(k\)-form.

From these definitions, we see that if \(\alpha_1, \ldots, \alpha_k \in V^*\), then we can iteratively define
\[
\alpha_1 \otimes \cdots \otimes \alpha_k := ((\alpha_1 \otimes \alpha_2) \otimes \alpha_3) \otimes \cdots \otimes \alpha_k
\]
and similar definitions for \(\alpha_1 \cdots \alpha_k\) and \(\alpha_1 \wedge \cdots \wedge \alpha_k\).

When we expand out the definitions of \(\alpha_1 \cdots \alpha_k\) and \(\alpha_1 \wedge \cdots \wedge \alpha_k\) there will be \(k!\) terms, one for each element of \(S_k\).

For any \(\sigma \in S_k\), we have
\[
\alpha_1 \cdots \alpha_k = \alpha_{\sigma(1)} \cdots \alpha_{\sigma(k)}
\]
and
\[
\alpha_1 \wedge \cdots \wedge \alpha_k = \varepsilon(\sigma)\alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(k)}
\]
The later property implies that \(\alpha_1 \wedge \cdots \wedge \alpha_k = 0\) if \(\alpha_i = \alpha_j\) for some \(i \neq j\).

The following result is helpful in understanding these iterated wedges.
Theorem 3.1. Let $\alpha_1, \ldots, \alpha_k \in V^*$.  

$$\alpha_1 \wedge \cdots \wedge \alpha_k = 0 \text{ if and only if } \alpha_1, \ldots, \alpha_k \text{ are linearly dependent}$$

Proof. Suppose that $\alpha_1, \ldots, \alpha_k$ is linearly dependent. Then there exists $x_1, \ldots, x_k$ such that  

$$x_1 \alpha_1 + \cdots + x_k \alpha_k = 0$$

and not all $x_1, \ldots, x_k$ are zero. Assume that $x_k \neq 0$. Let $H = \alpha_1 \wedge \cdots \wedge \alpha_{k-1}$ and let us apply $H \wedge$ to both sides of this equation. Using the above results and the linearity of $\wedge$, we deduce that  

$$x_k \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \alpha_k = 0$$

which implies that $\alpha_1 \wedge \cdots \wedge \alpha_k = 0$ as desired.

For the converse, suppose that $\alpha_1, \ldots, \alpha_k$ are linearly independent. Then we can extend $\alpha_1, \ldots, \alpha_k$ to a basis $\alpha_1, \ldots, \alpha_n$ for $V^*$. Let $v_1, \ldots, v_n$ be the dual basis for $V$. Then  

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \ldots, v_k) = 1$$

and so $\alpha_1 \wedge \cdots \wedge \alpha_k \neq 0$.  \hfill $\square$

The same method of proof can be used to prove the following result.

Theorem 3.2. Let $v_1, \ldots, v_k \in V$. Then there exists $H \in \Lambda^k V^*$ such that $H(v_1, \ldots, v_k) \neq 0$ if and only if $v_1, \ldots, v_k$ are linearly independent.

In particular this theorem shows that $\Lambda^k V^* = 0$ if $k > \dim V$.

### 3.3 Bases and dimension

We will now describe bases for our vector spaces of $k$-forms.

Theorem 3.3. Let $\alpha_1, \ldots, \alpha_n$ be a basis for $V^*$.

(i) $\{\alpha_{i_1} \otimes \cdots \otimes \alpha_{i_k}\}_{1 \leq i_1, \ldots, i_k \leq n}$ is a basis for $(V^*)^\otimes k$.  

(ii) $\{\alpha_{i_1} \cdots \alpha_{i_k}\}_{1 \leq i_1 \leq \cdots \leq i_k \leq n}$ is a basis for $\text{Sym}^k V^*$.  

(iii) $\{\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}\}_{1 \leq i_1 < \cdots < i_k \leq n}$ is a basis for $\Lambda^k V^*$.

Proof. We give the proof for the case of $(V^*)^\otimes k$ as the other cases are similar. So simplify the notation, let us assume that $k = 2$.

Let us first show that every bilinear form is a linear combination of $\{\alpha_i \otimes \alpha_j\}$. Let $H$ be a bilinear form. Let $v_1, \ldots, v_n$ be the basis of $V$ dual to $\alpha_1, \ldots, \alpha_n$. Let $c_{ij} = H(v_i, v_j)$ for each $i, j$. We claim that  

$$H = \sum_{i=1}^n \sum_{j=1}^n c_{ij} \alpha_i \otimes \alpha_j$$
Since both sides are bilinear forms, it suffices to check that they agree on all pairs \((v_k, v_l)\) of basis vectors. By definition \(H(v_k, v_l) = c_{kl}\). On the other hand,

\[
\left( \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \alpha_i \otimes \alpha_j \right)(v_k, v_l) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \alpha_i(v_k) \alpha_j(v_l) = c_{kl}
\]

and so the claim follows.

Now to see that \(\{\alpha_i \otimes \alpha_j\}\) is a linearly independent set, just note that if

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \alpha_i \otimes \alpha_j = 0,
\]

then by evaluating both sides on \((v_i, v_j)\), we see that \(c_{ij} = 0\) for all \(i, j\). \(\square\)

**Example 3.4.** Take \(n = 2, k = 2\). Then our bases are

\[\alpha_1 \otimes \alpha_1, \alpha_1 \otimes \alpha_2, \alpha_2 \otimes \alpha_1, \alpha_2 \otimes \alpha_2\]

and

\[\alpha_1 \cdot \alpha_1, \alpha_1 \cdot \alpha_2, \alpha_2 \cdot \alpha_2\]

and

\[\alpha_1 \wedge \alpha_2\]

**Corollary 3.5.** The dimension of \((V^*)^\otimes k\) is \(n^k\), the dimension of \(\text{Sym}^k V^*\) is \(\binom{n+k-1}{k}\) and the dimension of \(\Lambda^k V^*\) is \(\binom{n}{k}\).