1 Symmetric bilinear forms

We will now assume that the characteristic of our field is not 2 (so $1 + 1 \neq 0$).

1.1 Quadratic forms

Let $H$ be a symmetric bilinear form on a vector space $V$. Then $H$ gives us a function $Q : V \to F$ defined by $Q(v) = H(v, v)$. $Q$ is called a quadratic form. We can recover $H$ from $Q$ via the equation

$$H(v, w) = \frac{1}{2}(Q(v + w) - Q(v) - Q(w))$$

Quadratic forms are actually quite familiar objects.

**Proposition 1.1.** Let $V = F^n$. Let $Q$ be a quadratic form on $F^n$. Then $Q(x_1, \ldots, x_n)$ is a polynomial in $n$ variables where each term has degree 2. Conversely, every such polynomial is a quadratic form.

**Proof.** Let $Q$ be a quadratic form. Then

$$Q(x_1, \ldots, x_n) = [x_1 \cdots x_n]A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

for some symmetric matrix $A$.

Expanding this out, we see that

$$Q(x_1, \ldots, x_n) = \sum_{1 \leq i, j \leq n} A_{ij}x_ix_j$$

and so it is a polynomial with each term of degree 2. Conversely, any polynomial of degree 2 can be written in this form.

**Example 1.2.** Consider the polynomial $x^2 + 4xy + 3y^2$. This the quadratic form coming from the bilinear form $H_A$ defined by the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. 

We can use this knowledge to understand the graph of solutions to \( x^2 + 4xy + 3y^2 = 1 \). Note that \( H_A \) has a diagonal matrix \[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\] with respect to the basis \((1, 0), (-2, 1)\). This shows that \( Q(a(1, 0) + b(-2, 1)) = a^2 - b^2 \). Thus the solutions of \( x^2 + 4xy + 3y^2 = 1 \) are obtained from the solutions to \( a^2 - b^2 = 1 \) by a linear transformation. Thus the graph is a hyperbola.

### 1.2 Diagonalization

As we saw before, the bilinear form is symmetric if and only if it is represented by a symmetric matrix. We now will consider the problem of finding a basis for which the matrix is diagonal. We say that a bilinear form is diagonalizable if there exists a basis for \( V \) for which \( H \) is represented by a diagonal matrix.

**Lemma 1.3.** Let \( H \) be a non-trivial bilinear form on a vector space \( V \). Then there exists \( v \in V \) such that \( H(v, v) \neq 0 \).

**Proof.** There exist \( u, w \in V \) such that \( H(u, w) \neq 0 \). If \( H(u, u) \neq 0 \) or \( H(w, w) \neq 0 \), then we are done. So we assume that both \( u, w \) are isotropic. Let \( v = u + w \). Then \( H(v, v) = 2H(u, w) \neq 0 \).

**Theorem 1.4.** Let \( H \) be a symmetric bilinear form on a vector space \( V \). Then \( H \) is diagonalizable.

This means that there exists a basis \( v_1, \ldots, v_n \) for \( V \) for which \( [H]_{v_1,\ldots,v_n} \) is diagonal, or equivalently that \( H(v_i, v_j) = 0 \) if \( i \neq j \).

**Proof.** We proceed by induction on the dimension of the vector space \( V \). The base case is \( \dim V = 0 \), which is immediate. Assume the result holds for all bilinear forms on vector spaces of dimension \( n - 1 \) and let \( V \) be a vector space of dimension \( n \).

If \( H = 0 \), then we are already done. Assume \( H \neq 0 \), then by the Lemma we get \( v \in V \) such that \( H(v, v) \neq 0 \).

Let \( W = \text{span}(v) \perp \). Since \( v \) is not isotropic, \( W \oplus \text{span}(v) = V \). Since \( \dim W = n - 1 \), the result holds for \( W \). So pick a basis \( v_1, \ldots, v_{n-1} \) for \( W \) for which \( H_W \) is diagonal and then extend to a basis \( v_1, \ldots, v_{n-1}, v \) for \( V \). Since \( v_i \in W, H(v_i, v_i) = 0 \) for \( i = 1, \ldots, n-1 \). Thus the matrix for \( H \) is diagonal.

### 1.3 Diagonalization in the real case

For this section we will mostly work with real vector spaces. Recall that a symmetric bilinear form \( H \) on a real vector space \( V \) is called positive definite if \( H(v, v) > 0 \) for all \( v \in V, v \neq 0 \). A postive-definite symmetric bilinear form is the same thing as an inner product on \( V \).

**Theorem 1.5.** Let \( H \) be a symmetric bilinear form on a real vector space \( V \). There exists a basis \( v_1, \ldots, v_n \) for \( V \) such that \( [H]_{v_1,\ldots,v_n} \) is diagonal and all the entries are \( 1, -1, \) or \( 0 \).
We have already seen a special case of this theorem. Recall that if $H$ is an inner product, then there is an orthonormal basis for $H$. This is the same as a basis for which the matrix for $H$ consists of just 1s on the diagonal.

**Proof.** By the previous theorem, we can find a basis $w_1, \ldots, w_n$ for $V$ such that $H(w_i, w_j) = 0$ for $i \neq j$. Let $a_i = H(w_i, w_i)$ for $i = 1, \ldots, n$. Define

$$v_i = \begin{cases} \sqrt{a_i} w_i, & \text{if } a_i > 0 \\ \frac{1}{\sqrt{-a_i}} w_i, & \text{if } a_i < 0 \\ w_i, & \text{if } a_i = 0 \end{cases}$$

Then $H(v_i, v_i)$ is either 1, $-1$, or 0 depending on the three cases above. Also $H(v_i, v_j) = 0$ for $i \neq j$ and so we have found the desired basis. \hfill $\square$

**Corollary 1.6.** Let $Q$ be a quadratic form on a vector space $V$. There exists a basis $v_1, \ldots, v_n$ for $V$ such that the quadratic form is given by

$$Q(x_1v_1 + \cdots + x_nv_n) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

**Proof.** Let $H$ be a associated bilinear form. Pick a basis $v_1, \ldots, v_n$ as in the theorem, ordered so that the diagonal entries in the matrix are 1s then $-1$s, then 0s. The result follows. \hfill $\square$

Given a symmetric bilinear form $H$ on a real vector space $V$, pick a basis $v_1, \ldots, v_n$ for $V$ as above. Let $p$ be the number of 1s and $q$ be the number of $-1$s in the diagonal entries of the matrix. The following result is known (for some reason) as “Sylvester’s Law of Inertia”.

**Theorem 1.7.** The numbers $p, q$ depend only on the bilinear form. (They do not depend on the choice of basis $v_1, \ldots, v_n$.)

To prove this result, we will begin with the following discussion which applies to symmetric bilinear forms over any field. Given a symmetric bilinear form $H$, we define its radical (sometimes also called kernel) to be

$$\text{rad}(H) = \{ w \in V : H(v, w) = 0 \text{ for all } v \in V \}$$

In other words, $\text{rad}(H) = V^\perp$. Another way of thinking about this is to say that $\text{rad}(H) = \text{null}(H^\#)$.

**Lemma 1.8.** Let $H$ be a symmetric bilinear form on a vector space $V$. Let $v_1, \ldots, v_n$ be a basis for $V$ and let $A = [H]_{v_1, \ldots, v_n}$. Then

$$\dim \text{rad}(H) = \dim V - \text{rank}(A)$$

**Proof.** Recall that $A$ is actually the matrix for the linear map $H^\#$. Hence $\text{rank}(A) = \text{rank}(H^\#)$. So the result follows by the rank-nullity theorem for $H^\#$. \hfill $\square$
Proof of Theorem 1.7. The lemma shows us that \( p + q \) is an invariant of \( H \). So it suffices to show that \( p \) is independent of the basis.

Let
\[
\tilde{p} = \max (\dim W : W \text{ is a subspace of } V \text{ and } H|_W \text{ is positive definite})
\]
Clearly, \( \tilde{p} \) is independent of the basis. We claim that \( p = \tilde{p} \).

Assume that our basis \( v_1, \ldots, v_n \) is ordered so that
\[
H(v_i, v_i) = 1 \text{ for } i = 1, \ldots, p,
\]
\[
H(v_i, v_i) = -1 \text{ for } i = p + 1, \ldots, p + q, \text{ and}
\]
\[
H(v_i, v_i) = 0 \text{ for } i = p + q + 1, \ldots, n
\]
Let \( W = \text{span}(v_1, \ldots, v_p) \). Then \( \dim W = p \) and so \( p \leq \tilde{p} \).

To see that \( \tilde{p} \leq p \), let \( \tilde{W} \) be a subspace of \( V \) such that \( H|_{\tilde{W}} \) is positive definite and \( \dim \tilde{W} = \tilde{p} \).

We claim that \( \tilde{W} \cap \text{span}(v_{p+1}, \ldots, v_n) = 0 \). Let \( v \in \tilde{W} \cap \text{span}(v_{p+1}, \ldots, v_n) \), \( v \neq 0 \). Then \( H(v, v) > 0 \) by the definition of \( \tilde{W} \). On the other hand, if \( v \in \text{span}(v_{p+1}, \ldots, v_n) \), then
\[
v = x_{p+1}v_{p+1} + \cdots + x_nv_n
\]
and so \( H(v, v) = -x_{p+1}^2 - \cdots - x_{p+q}^2 \leq 0 \). We get a contradiction. Hence \( \tilde{W} \cap \text{span}(v_{p+1}, \ldots, v_n) = 0 \).

This implies that
\[
\dim \tilde{W} + \dim \text{span}(v_{p+1}, \ldots, v_n) \leq n
\]
and so \( \tilde{p} \leq n - (n - p) = p \) as desired. \(\square\)

The pair \((p, q)\) is called the signature of the bilinear form \( H \). (Some authors use \( p - q \) for the signature.)

Example 1.9. Consider the bilinear form on \( \mathbb{R}^2 \) given by the matrix \[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
It has signature \((1, -1)\).

Example 1.10. In special relativity, symmetric bilinear forms of signature \((3, 1)\) are used.

In the complex case, the theory simplifies considerably.

Theorem 1.11. Let \( H \) be a symmetric bilinear form on a complex vector space \( V \). Then there exists a basis \( v_1, \ldots, v_n \) for \( V \) for which \( [H]_{v_1, \ldots, v_n} \) is a diagonal matrix with only 1s or 0s on the diagonal. The number of 0s is the dimension of the radical of \( H \).

Proof. We follow the proof of Theorem 1.5. We start with a basis \( w_1, \ldots, w_n \) for which the matrix of \( H \) is diagonal. Then for each \( i \) with \( H(w_i, w_i) \neq 0 \), we choose \( a_i \) such that \( a_i^2 = \frac{1}{H(w_i, w_i)} \). Such \( a_i \) exists, since we are working with complex numbers. Then we set \( v_i = a_iw_i \) as before. \(\square\)