A cactus group action on crystals

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Suppose:

\[ \mathfrak{g} = \text{finite-dimensional, complex, semisimple Lie algebra} \]
\[ \mathfrak{h} = \text{a Cartan subalgebra of } \mathfrak{g} \]
\[ \Lambda = \text{the weight lattice of } \mathfrak{g} \]
\[ W_\mathfrak{g} = \text{the Weyl group of } \mathfrak{g} \]
\[ I = \text{the Dynkin diagram of } \mathfrak{g} \]
\[ \{\alpha_i\}_{i \in I} = \text{the simple roots of } \mathfrak{g} \]
\[ \theta_J = \text{Dynkin diagram automorphism of } J \subset I \text{ defined using the longest element of the Weyl group of } \mathfrak{g}_J, w_0^J: \]

\[ \alpha_{\theta_J(j)} = -w_0^J \cdot \alpha_j \quad \forall j \in J \]
The cactus group

Definition

The **cactus group** $J_g$ corresponding to $g$ is defined by:

**Generators:** $s_J$ where $J \subset I$ is a connected Dynkin subdiagram of $I$.

**Relations:**

1. $s_J^2 = 1 \quad \forall J \subset I$
2. $s_J s_{J'} = s_{J'} s_J \quad \forall J, J' \subset I$ disjoint
3. $s_{J'} s_J = s_{\theta_J(J')} s_J \quad \forall J' \subset J \subset I$
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The **pure cactus group** $PJ_g$ is defined by:

$1 \to PJ_g \to J_g \to W_g \to 1$

$s_J \mapsto w_0^J$
Let $g = \mathfrak{sl}_n$ with the usual numbering of the Dynkin nodes:

\[
\begin{array}{cccc}
\circ & \circ & \cdots & \circ \\
1 & 2 & \cdots & n-1 \\
\end{array}
\]

Then $\theta_I(i) = n - i$ and for $J_n$ we have:
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**Generators:** $\{s_{p,q}\}_{1 \leq p < q \leq n}$ corresponding to the Dynkin subdiagram with nodes $p$ to $q - 1$. 
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Generators: \( \{ s_{p,q} \}_{1 \leq p < q \leq n} \) corresponding to the Dynkin subdiagram with nodes \( p \) to \( q - 1 \).

Relations:

1. \( s_{p,q}^2 = 1 \) \( \forall 1 \leq p < q \leq n \)
2. \( s_{p,q}s_{k,l} = s_{k,l}s_{p,q} \) if \( [p, q] \) and \( [k, l] \) are disjoint
3. \( s_{p,q}s_{k,l} = s_{q+p-l, q+p-k}s_{p,q} \) if \( [k, l] \subset [p, q] \)
Example

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$1 \rightarrow PJ_n \rightarrow J_n \rightarrow S_n \rightarrow 1$

$s_{p,q} \mapsto \overline{s_{p,q}}$
The Schützenberger involution

\( B(\lambda) \) - the g-crystal corresponding to the highest weight representation \( V(\lambda) \), with:

- \( b_\lambda, b^{\text{low}}_\lambda \) highest and lowest weight elements
- \( \text{wt} : B(\lambda) \to \Lambda \) weight map
- \( e_i, f_i : B(\lambda) \to B(\lambda) \cup \{0\} \) Kashiwara operators
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**Definition**

The **Schützenberger involution** $\xi_{\lambda} : B(\lambda) \to B(\lambda)$ is defined by, for all $b \in B(\lambda)$:

$$\xi_{\lambda}(b_{\lambda}) = b_{\lambda}^{\text{low}}$$
$$e_i \xi_{\lambda}(b) = \xi_{\lambda}(f_{\theta(i)}b), \quad f_i \xi_{\lambda}(b) = \xi_{\lambda}(e_{\theta(i)}b)$$
$$\text{wt}(\xi_{\lambda}(b)) = w_0 \cdot \text{wt}(b)$$

More generally, $\xi : B \to B$ for any $g$-crystal $B$ by applying $\xi_{\lambda}$ to each connected component $B(\lambda)$. 
Theorem

For any $g$-crystal $B$, we have an action of the cactus group $J_g$ on $B$, defined by:

$$s_J(b) = \xi_{B_J}(b)$$

for any $J \subset I$, $b \in B$, where $B_J$ is the restriction of $B$ to the subdiagram $J$ of $I$. 

Action on crystals
Let $g = \mathfrak{sl}_3$, then

$$J_3 = \langle s_1, s_2, s_{12} \mid s_1^2 = s_2^2 = s_{12}^2 = 1, s_1 s_{12} = s_{12} s_2 \rangle.$$

$B(\alpha_1 + \alpha_2) =$ the adjoint representation crystal:

The $B(\alpha_1 + \alpha_2)$ crystal.

The $s_{12}$ action.

The $s_1$ action.
Theorem (Davis-Januszkiewicz-Scott, 2002)

With the previous notation,

\[ PJ_g = \pi_1(\mathbb{P}(h^{\text{reg}}_\mathbb{R})) \]

In the case \( g = \mathfrak{sl}_n \), \( PJ_n = \pi_1(M_0^{n+1}(\mathbb{R})) \), where:

\[ M_0^{n+1}(\mathbb{R}) = \frac{(\mathbb{R}P^1)^{n+1} - \Delta)}{PGL_2(\mathbb{R})} \]

\( \Delta \) denotes the thick diagonal.
Consider $S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}]$ with the Poisson bracket $\{\cdot, \cdot\}$ defined by:

1. $\{x, y\} = [x, y] \quad \forall \ x, y \in \mathfrak{g}$
2. $\{fg, h\} = f \{g, h\} + \{f, h\} g \quad \forall \ f, g, h \in S(\mathfrak{g})$

**Theorem (Mishchenko-Fomenko ’78, Tarasov ’00, Vinberg ’91)**

For any $\mu \in \mathbb{P}(\mathfrak{h}^{\text{reg}})$, the shift of argument algebra

$$A_\mu = \langle F_i, \partial_\mu^n F_i \rangle$$

where $\{F_i\}_{i=1,\ldots,\text{rk}(\mathfrak{g})}$ is a set of algebraically independent generators of $Z(S(\mathfrak{g})) = S(\mathfrak{g})^\mathfrak{g}$, is a maximal Poisson-commutative subalgebra of $S(\mathfrak{g})$. 
Conjecture

(Known for \( \mathfrak{g} = \mathfrak{sl}_n \).)

1. The family \( \{ A_\mu \}_\mu \) admits a compactification and lifting \( \{ A_\mu \}_\mu \) to \( U(\mathfrak{g}) \).

2. For any \( \mu \in \overline{P(h_{\text{reg}}^R)} \) and any highest weight \( \mathfrak{g} \)-rep. \( V(\lambda) \), \( A_\mu \) acts on \( V(\lambda) \) with simple spectrum.

3. This induces a covering \( E \xrightarrow{\phi} (\overline{P(h_{\text{reg}}^R)}, \mu_0) \) and so a monodromy action \( \pi_1(\overline{P(h_{\text{reg}}^R)}, \mu_0) \acts \phi^{-1}(\mu_0) \).
Theorem

For \( \mathfrak{g} = \mathfrak{sl}_n \), under the monodromy action:

\[
PJ_\mathfrak{g} = \pi_1(\overline{\mathbb{P}(\mathfrak{h}_{\mathbb{R}}^{\text{reg}})}, \mu_0) \curvearrowright \phi^{-1}(\mu_0)
\]

The two sets \( \phi^{-1}(\mu_0) \cong B(\lambda) \) as \( PJ_\mathfrak{g} \) sets.

Partial results suggest this is true in general.
The End

Thank you!