Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a multiple-choice answer sheet, on which you should fill in your name, number, tutorial time, tutorial room, and tutor’s name.

This test consists of 10 multiple choice questions, and 4 written-answer questions. For the multiple choice questions you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter on the answer sheet with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the written-answer questions, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ____________________________

GIVEN NAME: ____________________________

STUDENT NO: ____________________________

SIGNATURE: ____________________________

TUTORIAL TIME and ROOM: ____________________________

RECODE and TIMECODE: ____________________________

T.A.'S NAME: ____________________________

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PART A. Multiple Choice

1. [4 marks]
If $-\frac{1}{2} \leq x \leq \frac{1}{2}$, the function $y = f(x) = \frac{1 + x^2}{1 - x^2}$ has

A. no absolute maximum value.
B. an absolute maximum value of $y = \frac{3\sqrt{3}}{2}$.
C. an absolute maximum value of $y = 1$.
D. an absolute maximum value of $y = \frac{2}{3}$.
E. an absolute maximum value of $y = \infty$.

\[
\frac{f'(x)}{f(x)} = \frac{2x(1-x^2) + 2x(1-x^2)}{(1-x^2)^2} = \frac{4x}{(1-x^2)^2}
\]
In $[-\frac{1}{2}, \frac{1}{2}]$, crit. at $x = 0$ only.
Cont. on $[-\frac{1}{2}, \frac{1}{2}]$, so must have absolute max
at $x = \frac{1}{2}, -\frac{1}{2}$ or $0$.

\[
f\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right) = \frac{1 + \frac{1}{4}}{1 - \frac{1}{4}} = \frac{5}{3}
\]
\[
f(0) = 1
\]
\[
\frac{5}{3} > 1 \quad \text{so} \quad y = \frac{5}{3} \quad (D)
\]

2. [4 marks]
The graph of the curve $y = x^3e^{-x}$ has

A. no points of inflection.
B. 1 point of inflection.
C. 2 points of inflection.
D. 3 points of inflection.
E. 4 points of inflection.

\[
y' = 3x^2e^{-x} - x^3e^{-x}
\]
\[
y'' = 6xe^{-x} - 3x^2e^{-x} - 3xe^{-x} + 2xe^{-x} = xe^{-x}(6 - 6x + x^2)
\]
\[
y'' = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 6 \pm \sqrt{36 - 4} = 3 \pm \sqrt{3}
\]

\[
\begin{array}{c|c|c|c|c|c}
(-\infty, 0) & + & down & x = -1 \\
(0, 3 - \sqrt{3}) & + & up & x = 1 \\
(3 - \sqrt{3}, 3 + \sqrt{3}) & - & down & x = 3 \\
(3 + \sqrt{3}, \infty) & + & up & 1000
\end{array}
\]
All 3 roots of $y''$ are p.o.i. (D)
3. [4 marks]

The marginal revenue function for a product is given by \( \frac{dr}{dq} = 1000 - 3q^2 \) where \( r \) is revenue in dollars and \( q \) is the number of units sold. Assuming that revenue is 0 if no units are sold, how many units will be sold if the unit price is $100?

A. 9

\[ r = 1000q - q^3 + K \]

\[ \begin{align*}
0 &= 1000(0) - 0^3 + K \\
K &= 0
\end{align*} \]

B. 33

\[ p = 1000 - q^2 \]

C. 30

\[ p = 1000 - q^2 \]

\[ 100 = 1000 - q^2 \]

\[ q^2 = 900 \]

\[ q = 30 \]

D. 8

E. 17

\( C \)

\[ q = 30 \]

4. [4 marks]

If \( \int_1^2 f(x) \, dx = -6 \) and \( \int_5^6 f(x) \, dx = 4 \), then \( \int_1^6 f(x) \, dx = \)

A. \(-10\)

\[ \int_1^5 = \int_1^2 + \int_2^5 = \int_1^2 - \int_5^2 \]

\[ = -6 - 4 \]

\[ = -10 \]

B. \(-2\)

C. \(2\)

D. \(10\)

E. \(-\frac{3}{2}\)
5. [4 marks]
\[ \int_{1}^{4} \sqrt{x} \, dx = \frac{1}{2} \int_{1}^{4} \, dx = \frac{2}{3} \times 3^{3/2} \Bigg|_{1}^{4} = \frac{2}{3} \left[ 4^{3/2} - 1 \right] = \frac{2}{3} \cdot 7 = \frac{14}{3} \] 
A. \( \frac{21}{2} \)
B. \( \frac{7}{3} \)
C. 1
D. -1
E. \( \frac{14}{3} \)

6. [4 marks]
If \( f(x) = \int_{0}^{x} e^{\sqrt{3t^2 + 1}} \, dt \), then \( f'(8) = \)
A. \( e^8 - e \)
B. \( \frac{3}{16} (e^8 - e) \)
C. \( \frac{3}{16} e^8 \)
D. \( 3e^8 \)
E. \( e^8 \)
7. [4 marks]
If demand is given by \( p = 10 - q \) and supply is given by \( p = q + 2 \), the consumers' surplus (CS) is equal to:

A. 18
B. 16
C. 32
D. 24
E. 8

\[
CS = \int_{0}^{10-4} \left(10 - x - (2 + x)\right) \, dx = \int_{0}^{4} [8 - 2x] \, dx = 8 \]  (E)

or more simply, the area of the CS triangle is
\[
\frac{1}{2} \cdot 4 \cdot 4 = 8
\]

8. [4 marks]
The average value of \( f(x) = x^{-2} \) on the interval \([\frac{1}{3}, 1]\) is

A. \( \frac{36}{7} \)
B. 7
C. \( \frac{28}{3} \)
D. 6
E. 8

\[
\bar{f} = \frac{1}{1 - \frac{1}{3}} \int_{\frac{1}{3}}^{1} \frac{1}{x^2} \, dx = \frac{3}{2} \left( -1 - \left( -1 \right) \right) = \frac{3}{2} \cdot 6 = 9 \]  (B)
9. [4 marks]
A continuous cash-flow of $10,000 per year for 10 years at 4% per year compounded continuously has a present value closest to

A. $67,032
B. $81,109
C. $82,420
D. $122,956
E. $167,580

\[ PV = \int_0^{10} 10,000 e^{-0.04t} \, dt \]
\[ = \left. \frac{10,000}{-0.04} e^{-0.04t} \right|_0^{10} \]
\[ = 250,000 (1 - e^{-0.4}) \]
\[ \approx 82,419.99 \quad \text{(C)} \]

10. [4 marks]
\[ \int_{-1}^{0} \left( \frac{1}{x+2} + 2^x \right) \, dx = \left[ \ln|x+2| + \frac{2^x}{\ln 2} \right]_{-1} \]
A. $\ln 2 + \frac{1}{2 \ln 2}$
B. $\frac{3}{2} \ln 2$
C. $1 + \frac{1}{2 \ln 2}$
D. $\ln 2 + \frac{1}{\ln 2}$
E. $1 + 2 \ln 2$
PART B. Written-Answer Questions

1. [16 marks]

Given: \( f(x) = \frac{e^x}{x^3} \)

\( f'(x) = e^x x^{-4}(x - 3) \)

and \( f''(x) = e^x x^{-5}(x^2 - 6x + 12) \)

[3] (a) find and justify all vertical and horizontal asymptotes.

\[
\begin{align*}
\lim_{x \to 0^+} \frac{e^x}{x^3} & = +\infty & \text{V.A. at } x = 0 \\
\lim_{x \to 0^-} \frac{e^x}{x^3} & = -\infty
\end{align*}
\]

H.A.: \( \lim_{x \to \infty} \frac{e^x}{x^3} = 0 \quad \lim_{x \to -\infty} \frac{e^x}{x^3} = 0 \)

\[
\begin{align*}
H.A. & : & x = 0 \\
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to \infty} \frac{e^x}{x^3} & = 0 \\
\lim_{x \to -\infty} \frac{e^x}{x^3} & = 0
\end{align*}
\]

[3] (b) find where \( f \) is increasing and decreasing and any relative maxima and/or minima (with explanation).

\[
\begin{array}{c|c|c}
\text{Interval} & \text{Increasing} & \text{Decreasing} \\
\hline
(-\infty, 0) & & \\
(0, 3) & & \\
(3, \infty) & & \\
\end{array}
\]

\( f \) is increasing on \( (3, \infty) \) and decreasing on \( (-\infty, 0) \) and \( (0, 3) \).

\( f \) has a local min at \( x = 3 \).

[3] (c) find where \( f \) is concave upward and concave downward and any inflection points (with explanation).

\( x^2 - 6x + 12 \) has no real roots \( (6^2 - 4 \cdot 12 < 0) \).

\( f'' \) is never 0 and only fails to exist at \( x = 0 \) which is not a point on the curve.

\[
\begin{align*}
\text{No p.o.i.} & \\
\text{Down} & (-\infty, 0) \quad \text{Up} \quad (0, \infty) \\
\end{align*}
\]

[7] (d) graph \( y = f(x) \) clearly on the axes below.
2. [14 marks]

A company finds that to sell and produce \( q \) units of its product it must set its unit price (to the customer) at \( 100e^{-0.01q} \) dollars, while it costs a total of \( 400 - 300e^{-0.01q} \) dollars to produce the \( q \) units.

[7] (a) Find the number of units the company should produce to maximize its profit.

\[
\pi = pq - c = pq - \left( 100e^{-0.01q} \right) \left( 400 - 300e^{-0.01q} \right)
\]

\[
\frac{d\pi}{dq} = 100e^{-0.01q} - 300e^{-0.01q} - q(-0.01) \cdot 100e^{-0.01q} - 3e^{-0.01q}
\]

\[
= e^{-0.01q}(97 - q) = 0 \quad \text{when} \quad q = 97
\]

\[
0 < q
\]

[8] (b) Justify your answer to (a).

Note that \( \frac{d\pi}{dq} > 0 \) when \( q < 97 \), so \( \pi \) is increasing from \( q = 0 \) to \( q = 97 \).

but \( \frac{d\pi}{dq} < 0 \) for \( q > 97 \), so \( \pi \) is decreasing for ever afterwards.

While it is true that \( \frac{d^2\pi}{dq^2} < 0 \) at \( q = 97 \), this only implies a local max. (and for \( q \) big enough, the profit curve becomes concave up).

[9] (c) To the nearest dollar, what unit price should the company charge its customers to maximize its profit?

\[
p = 100e^{-0.97} \approx \$37.91
\]

\[
\$38
\]
3. [14 marks]
Write the area of the region(s) between \( y = \frac{5}{x} \) and \( y = 6 - x \) from \( x = 1 \) to \( x = 6 \) as (a) definite integral(s) and evaluate the integral(s). (Leave your answer in exact form rather than as a decimal number.)

\[
\int_{1}^{6} \left[ \frac{5}{x} - (6-x) \right] dx
\]

\[
= \left[ 5 \ln|x| - 6x + \frac{x^2}{2} \right]_{1}^{6}
\]

\[
= \left[ 5 \ln 6 - 36 + 18 \right] - \left[ 5 \ln 1 - 6 + \frac{1}{2} \right]
\]

\[
= \left[ 30 - 25 \right] + \left[ \frac{1}{2} + 5 \ln 6 - 5 \ln 5 \right]
\]

\[
= \frac{23}{2} - 10 \ln 5 + 5 \ln 6
\]
4. [16 marks] Evaluate the following integrals

\[ \int_0^2 xe^{x^2} \, dx \]
\[ u = x, \quad dv = e^{x^2} \, dx \]
\[ du = dx, \quad v = 2e^{x^2} \]
\[ = 2xe^{x^2}\bigg|_0^2 - \int_0^2 2e^{x^2} \, dx \]
\[ = (4e-0) - 4e^{x^2}\bigg|_0^2 \]
\[ = 4e - 4(e-1) = 4 \]

\[ \int \frac{x}{2x + 3} \, dx \]
\[ \frac{1}{2x + 3} \frac{x}{x + \frac{3}{2}} \]
\[ = \frac{1}{2} \ln|2x+3| + C \]
Alternatively:
\[ u = 2x + 3, \quad du = 2 \, dx \]
\[ \int \frac{u-3}{2u} \, du = \frac{1}{4} \left( \ln|2x+3| + C \right) \]
\[ = \frac{1}{4} \left( 2x + 3 \right) - \frac{3}{4} \ln|2x+3| + C \]
\[ \frac{11 - 3x}{x^2 + x - 6} \]
\[ = \frac{A}{x + 3} + \frac{B}{x - 2} \]
\[ A(x-2) + B(x+3) = 11 - 3x \]
\[ x = 2, \quad -5A = 20, \quad A = -4 \]
\[ x = -3, \quad 5B = 5, \quad B = 1 \]
\[ \int \frac{11 - 3x}{x^2 + x - 6} \, dx = \int \left( -\frac{4}{x+3} + \frac{1}{x-2} \right) \, dx \]
\[ = -4 \ln|x+3| + \ln|x-2| + C \]