Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a multiple-choice answer sheet, on which you should fill in your name, number, tutorial time, tutorial room, and tutor’s name.

This test consists of 10 multiple choice questions, and 4 written-answer questions. For the multiple choice questions you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter on the answer sheet with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the written-answer questions, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ____________________________

GIVEN NAME: ____________________________

STUDENT NO: ____________________________

SIGNATURE: ____________________________

TUTORIAL TIME and ROOM: ____________________________

REGCODE and TIMECODE: ____________________________

T.A.’S NAME: ____________________________

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FOR MARKER ONLY

Multiple Choice

B1
B2
B3
B4
TOTAL
PART A. Multiple Choice

1. \(4\) marks

If \(f\) is defined for all real \(x\) except \(x = 2\) by

\[
f(x) = \begin{cases} 
1 & \text{if } x \leq 0 \\
0 & \text{if } 0 < x \leq 1 \\
\frac{1}{2-x} & \text{if } x > 1 
\end{cases}
\]

then \(f\) is not continuous at

A. \(x = 1\) and \(x = 2\) only
B. \(x = 0\) only
C. \(x = 0\) and \(x = 2\) only
D. \(x = 1\) only
E. \(x = 0, x = 1,\) and \(x = 2\) only

The only candidates are \(x = 0, x = 1\) and \(x = 2\).

\[
\lim_{x \to 0^-} f(x) = 1 \neq \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0
\]

\[
\begin{align*}
\text{discont at } & x = 0 \\
\lim_{x \to 1^-} f(x) & = \lim_{x \to 1^-} x = 1 \\
\lim_{x \to 1^+} f(x) & = \lim_{x \to 1^+} \frac{1}{2-x} = 1 \\
\text{cont at } & x = 1 \\
\text{discont at } & x = 2
\end{align*}
\]

(\(C\))

2. \(4\) marks

If \(y = \frac{x}{\ln(x)}\), then when \(x = e^2\), \(\frac{dy}{dx} = \frac{\ln(x) - 1}{x \left(\ln(x)\right)^2} = \frac{\ln(x) - 1}{(\ln(x))^2}
\]

\[
= \frac{\ln(e^2) - 1}{(\ln(e^2))^2} = \frac{2 - 1}{2^2} = \frac{1}{4}
\]

A. \(\frac{e}{2}\)
B. \(\frac{1}{4}\)
C. \(\frac{e^2}{2}\)
D. 0
E. \(\frac{1}{2}\)

(\(B\))
3. [4 marks]
If \( f(x) = x^2 e^{3x} \), then \( f''(1) = \)

A. \( 9e^3 \)
B. \( 4e^3 \)
C. \( 36e^3 \)
D. 2
E. \( 23e^3 \)

\[
\begin{align*}
  f'(x) &= 2xe^{3x} + 3x^2 e^{3x} \\
  f''(x) &= 6xe^{3x} + 6xe^{3x} + 9x^2 e^{3x} \\
  &= e^{3x}(2 + 12x + 9x^2)
\end{align*}
\]

\( f''(1) = 23e^3 \) \( \boxed{E} \)

4. [4 marks]
Let \( h(x) = f(x) \cdot g(x) \), where

\[
\begin{align*}
  f(1) &= 3 & f(5) &= 4 & g(1) &= 5 \\
  f'(1) &= 2 & f'(5) &= 9 & g'(1) &= 7
\end{align*}
\]
Then \( h'(1) = \)

A. 31
B. 63
C. 29
D. 73
E. 30

\[
\begin{align*}
  h'(x) &= f'(x)g(x) + f(x)g'(x) \\
  h'(1) &= f'(1)g(1) + f(1)g'(1) \\
  &= 2 \cdot 5 + 3 \cdot 7 \\
  &= 31 \boxed{A}
\end{align*}
\]
5. [4 marks]
Let \( h(x) = \sqrt{f(x)} \), where
\[
\begin{align*}
  f(0) &= 25 & f(1) &= 0 \\
  f'(0) &= 4 & f'(1) &= 9
\end{align*}
\]
Then \( h'(0) = \]

A. 2  
B. 0  
C. \( \frac{2}{3} \)  
D. \( \frac{2}{5} \)  
E. \( \frac{9}{4} \)

6. [4 marks]
The Consumption Function is given by: \( C = 5 + I - 4\sqrt{I} \). If \( I \) is income and \( S \) is savings, which of the following is false?

A. The marginal propensity to consume = \( \frac{\sqrt{I} - 2}{\sqrt{I}} \)  
B. \( S + C = I \) true, by definition  
C. The rate of change of savings with respect to \( I \) is \( \frac{2}{\sqrt{I}} \) when \( I = 16 \)  
D. \( \frac{dS}{dI} = \frac{dC}{dI} \) when \( I = 16 \)  
E. \( S = 5 - 4\sqrt{I} \) True

\[
\begin{align*}
  \text{MPC} &= \frac{dC}{dI} = 1 - \frac{2}{\sqrt{I}} = \frac{\sqrt{I} - 2}{\sqrt{I}} \text{ true} \\
  \frac{dS}{dI} &= \frac{dC}{dI} = \frac{2}{\sqrt{I}} \text{ true} \\
  \text{When } I &= 16, \quad 1 - \frac{2}{\sqrt{16}} = 1 - \frac{1}{2} = \frac{1}{2} \text{ and } \frac{2}{\sqrt{16}} = \frac{2}{4} = \frac{1}{2} \text{ so true} \\
  S &= I - C = I - (5 + I - 4\sqrt{I}) \\
  &= 4\sqrt{I} - 5 \text{ and } S = 5 - 4\sqrt{I}.
\end{align*}
\]
7. \[4\text{ marks}\]

If Newton's method is used to approximate a zero of \(f(x) = x^5 + x + 1\) and the initial approximation is \(x_1 = 0\), then \(x_3 =\)

A. \(-2\)

\[x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\]

B. \(5\)

C. \(-1\)

\[x_{n+1} = x_n - \frac{x_n^5 + x_n + 1}{5x_n^4 + 1}\]

D. \(-\frac{5}{6}\)

E. \(-\frac{1}{2}\)

If \(x_1 = 0\), \(x_2 = -1\)

\[x_3 = \frac{4(-1)^5 - 1}{5(-1)^4 + 1} = -\frac{5}{6}\]

\(\text{(D)}\)

8. \[4\text{ marks}\]

\[
\lim_{x \to 1} \frac{(x-1)^2}{e^{x-1} - x} = \lim_{x \to 1} \frac{2(x-1)}{e^{x-1} - 1} = \lim_{x \to 1} \frac{2}{e^{x-1}} = 2 \text{ by substitution}\]

A. \(e^{-1}\)

B. \(\infty\)

C. \(2\)

D. \(0\)

E. \(1\)
9. [4 marks]

\[ \lim_{x \to 0} \frac{e^x}{x} - \frac{1}{xe^x} = \infty - \infty \]

A. \( \infty \)

\[
\lim_{x \to 0} \frac{e^x - 1}{e^x} = 0
\]

B. 0

C. \( e \)

\[
\lim_{x \to 0} \frac{2e^{2x}}{e^x + xe^x} = 2
\]

D. 2

E. 1

or:

\[
\frac{e^x - 1}{xe^x} = \frac{e^x - e^{-x}}{x}
\]

\[
\lim_{x \to 0} \frac{e^x - e^{-x}}{x} = \lim_{x \to 0} \frac{e^x + e^{-x}}{1} = 2 \text{ as before}
\]

10. [4 marks]

The asymptotes of \( f(x) = xe^{-x} \) are:

A. \( x = 0 \) and \( y = 0 \) 

\[ P \text{ is cont everywhere so}
\]

\[ \text{has no V.A.} \]

\[ \lim_{x \to -\infty} f(x) = -\infty \]

B. \( x = 0 \) and \( y = 1 \)

but \( \lim_{x \to \infty} f(x) = \lim_{x \to 0} \frac{x}{e^x} \)

\[ = \lim_{x \to 0} \frac{1}{e^x} = 0 \]

\[ \text{so } y = 0 \text{ is H.A.} \]

C. \( y = 0 \) only

D. \( y = 1 \) only

E. \( f \) has no asymptotes
1. [11 marks]

Solve the inequality \( \frac{x - 3}{\ln x - 2} \geq 0 \).

\( \frac{x - 3}{\ln x - 2} \) is only defined for \( x > 0 \).

It is cont. on \( x > 0 \) as long as \( \ln x \neq 2 \), that is \( x \neq e^2 \).

Hence \( \frac{x - 3}{\ln x - 2} \) is cont. and unequal to zero, for \( x > 0 \), except at \( x = 3 \) and \( x = e^2 > 3 \).

\[
\begin{array}{|c|c|}
\hline
(0, 3) & + \\
\hline
(3, e^2) & - \\
\hline
(e^2, \infty) & + \\
\hline
\end{array}
\]

Inside each of these intervals \( f \) is cont. and never zero, hence \( f \) cannot change sign. A single test point will do.

In \((0, 3)\) try \(1\):

\[
\frac{1 - 3}{\ln 1 - 2} = -\frac{2}{-1} = 2 > 0
\]

In \((3, e^2)\) try \(4\): either by calculator or by calculator or by \(4 < e^2\), the denominator < 0 and \(x - 3 > 0\).

In \((e^2, \infty)\), numerator and denominator obviously > 0.

The solution is

\[(0, 3] \cup (e^2, \infty)\]

or: \(0 < x \leq 3 \) or \( e^2 < x \)
2. [16 marks]

If the revenue function is given by: \( r = \frac{3000q}{q + 100} \) and \( q = 100m - m^2 \) where \( m \) is the number of employees needed to produce \( q \) units, find

[4] (a) the marginal revenue when \( q = 900 \).

\[
MR = \frac{d}{dq} \left[ \frac{3000q}{q + 100} \right] = \frac{3 \times 10^5}{(q + 100)^2}
\]

\[
\frac{d}{dq} = \frac{3 \times 10^5}{10000} = \frac{3}{10} = 0.3 \quad \text{at } q = 900
\]

[4] (b) the marginal revenue product when \( m = 10 \). When \( m = 10 \), \( q = 1000 - 100 \)

\[
\frac{d}{dm} = \frac{d}{dq} \frac{dq}{dm}
\]

\[
= 0.3 \quad \text{when } m = 10, \quad \frac{dq}{dm} = 100 - 2m
\]

\[
= 0.3 \times 80 = 24
\]

[4] (c) elasticity of demand when \( q = 900 \).

\[
pq = r = \frac{3000q}{q + 100}
\]

\[
\eta = \left| \frac{d}{dq} \right| = \frac{p}{q} \frac{dq}{dp} = \frac{p}{q} \left( \frac{-3000}{(q + 100)^2} \right) = -\frac{3000}{q + 100} \cdot \frac{q}{(q + 100)^2} = -\frac{3000}{q + 100} \cdot \frac{q}{(q + 100)}
\]

\[
= -\frac{1000}{900} = -\frac{10}{9}
\]

[4] (d) Is demand elastic or inelastic at \( q = 900 \)? For what values of \( q \) is demand elastic?

Since \( |\eta| = \frac{10}{9} > 1 \) demand is elastic at \( q = 900 \)

Since \( |\eta| = \frac{q + 100}{q} > 1 \) for every \( q \)

demand is always elastic
3. [17 marks]

[6] (a) If \( y = \frac{2^{x^2}(x - 4)^2}{e^{x^3 + 1}} \), find \( \frac{dy}{dx} \) in terms of \( x \) only.

\[
\frac{dy}{dx} = \frac{2^{x^2} (x - 4)^2 e^{x^3 + 1}}{e^{x^3 + 1}} \left[ 2x \ln 2 + 2 \ln (x - 4) + \frac{2x}{x - 4} + 1 + \frac{12x^2}{x^3 + 1} \right]
\]

(b) (For part (b), simplify your answers as much as possible.)

If \( y^3 = e^{x-y} \) defines \( y \) implicitly as a function of \( x \),

[6] (i) find \( \frac{dy}{dx} \) in terms of \( y \) only.

\[
3y \frac{dy}{dx} = e^{x-y} \left( 1 - \frac{dy}{dx} \right)
\]

\[
\frac{dy}{dx} = \frac{e^{x-y}}{3y + e^{x-y}} = \frac{y^3}{3y^2 + y}
\]

[5] (ii) With the same \( y \), find \( \frac{d^2y}{dx^2} \) in terms of \( y \) only.

\[
\frac{d^2y}{dx^2} = \frac{(3+y) \frac{dy}{dx} - y \frac{dy}{dx}}{(3+y)^2} = \frac{3 \frac{dy}{dx}}{(3+y)^2}
\]

\[
= \frac{3y}{(3+y)^3}
\]
4. [16 marks]

[8] (a) Recall that the relative rate of change of \( y = f(x) \) is \( \frac{1}{y} \frac{dy}{dx} \).

Given the total cost function \( c = q(1 + \frac{1}{q})^q \),

find the relative rate of change of **average** cost when \( q = 10 \). (Your answer should be a single number to 4 decimal places.)

\[
\text{relative rate of change} = \frac{1}{f} \frac{df}{dx} = \frac{d}{dx} \left( \ln c \right)
\]

\[
\ln c = \frac{c}{q} = (1 + \frac{1}{q})^q
\]

\[
\frac{d}{dq} \left( \ln c \right) = \frac{d}{dq} \left[ q \ln \left( 1 + \frac{1}{q} \right) \right]
\]

\[
= \ln \left( 1 + \frac{1}{q} \right) + \frac{1}{1 + \frac{1}{q}} \cdot \frac{-1}{q^2}
\]

\[
= \ln \left( 1 + \frac{1}{q} \right) - \frac{q}{1 + \frac{1}{q}}
\]

\[
= \ln 1.1 - \frac{10}{1 + \frac{1}{10}} \approx 0.0044
\]

[8] (b) If \( f(x) = (2x)^{ln x} \) when \( x > 0 \), find all critical points of \( y = f(x) \) with \( x > 0 \).
(Do not classify.)

\[
\ln f = \ln x \cdot \ln (2x)
\]

\[
\frac{1}{f} f' = \frac{1}{x} \ln 2x + \frac{\ln x}{x}
\]

\[
= \frac{\ln 2x + \ln x}{x} = \frac{\ln 2 + \ln x}{x}
\]

\[
f' = (2x)^{\ln x} \cdot \frac{(\ln 2 + \ln x)}{x}
\]

For \( x > 0 \), critical only when \( f' = 0 \)

\[
i.e. \text{ when } \ln x = -\frac{1}{2} \ln 2 = \ln \left( \frac{1}{\sqrt{2}} \right)
\]

\[
i.e. \ x = \frac{1}{\sqrt{2}}
\]