Department of Mathematics  
University of Toronto  
Tuesday, January 17, 2012, 6:10 - 8:00 PM  
MAT 133Y  TERM TEST #2  
Calculus and Linear Algebra for Commerce  
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a multiple-choice answer sheet, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple-choice questions, and 4 written-answer questions.

For the multiple choice questions you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter on the answer sheet with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the written-answer questions, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: 

GIVEN NAME: 

STUDENT NO: 

SIGNATURE: 

TUTORIAL TIME and ROOM: 

REGCODE and TIMECODE: 

T.A.'S NAME: 

---

<table>
<thead>
<tr>
<th>Regcode</th>
<th>Timeslot</th>
<th>Room</th>
<th>Regcode</th>
<th>Timeslot</th>
<th>Room</th>
</tr>
</thead>
<tbody>
<tr>
<td>T0101A</td>
<td>M9A</td>
<td>SS102</td>
<td>T0601A</td>
<td>R4A</td>
<td>RV142</td>
</tr>
<tr>
<td>T0101B</td>
<td>M9B</td>
<td>SS107</td>
<td>T0601B</td>
<td>R4B</td>
<td>GR248</td>
</tr>
<tr>
<td>T0101C</td>
<td>M9C</td>
<td>SS108</td>
<td>T0601C</td>
<td>R4C</td>
<td>AB107</td>
</tr>
<tr>
<td>T0201A</td>
<td>M3A</td>
<td>SS210</td>
<td>T0601D</td>
<td>R4D</td>
<td>GB221</td>
</tr>
<tr>
<td>T0201B</td>
<td>M3B</td>
<td>MP134</td>
<td>T0701A</td>
<td>F2A</td>
<td>LM156</td>
</tr>
<tr>
<td>T0201C</td>
<td>M3C</td>
<td>RW145</td>
<td>T0701B</td>
<td>F2B</td>
<td>RV229</td>
</tr>
<tr>
<td>T0201D</td>
<td>M3D</td>
<td>UC228</td>
<td>T0701C</td>
<td>F2C</td>
<td>BA1240</td>
</tr>
<tr>
<td>T0301A</td>
<td>T3A</td>
<td>RW229</td>
<td>T0701D</td>
<td>F2D</td>
<td>MS4779</td>
</tr>
<tr>
<td>T0301B</td>
<td>T3B</td>
<td>WS344</td>
<td>T0801A</td>
<td>F3A</td>
<td>SE1982</td>
</tr>
<tr>
<td>T0301C</td>
<td>T3C</td>
<td>MP137</td>
<td>T0801B</td>
<td>F3B</td>
<td>SS2110</td>
</tr>
<tr>
<td>T0401A</td>
<td>W9A</td>
<td>SS107</td>
<td>T0801C</td>
<td>F3C</td>
<td>W1323</td>
</tr>
<tr>
<td>T0401B</td>
<td>W9B</td>
<td>SS107</td>
<td>T0901A</td>
<td>M5A</td>
<td>SE1987</td>
</tr>
<tr>
<td>T0501A</td>
<td>W3A</td>
<td>SS125</td>
<td>T0901B</td>
<td>M5B</td>
<td>MP1244</td>
</tr>
<tr>
<td>T0501B</td>
<td>W3B</td>
<td>WS525</td>
<td>T0901C</td>
<td>M5C</td>
<td>MP137</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

FOR MARKER ONLY

- Multiple Choice

<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
</tr>
<tr>
<td>B2</td>
</tr>
<tr>
<td>B3</td>
</tr>
<tr>
<td>B4</td>
</tr>
<tr>
<td>TOTAL</td>
</tr>
</tbody>
</table>

Page 1 of 10
PART A. Multiple Choice

1. \( \frac{d}{dx} \left( \frac{x-1}{x+2} \right) > 0 \) then
   
   A. \( x < -2 \text{ or } 0 < x < 1 \)
   B. \( -2 < x \leq 0 \text{ or } x > 1 \)
   C. \( x < -2 \text{ or } 0 \leq x \leq 1 \)
   D. \( -2 < x \leq 0 \text{ or } x > 1 \)
   E. \( -2 < x < 0 \text{ or } x > 1 \)

2. \( \int_1^2 x^2 \, dx \)
   
   A. \( -\frac{1}{3} \)
   B. \( -\frac{10}{9} \)
   C. \( -\frac{2}{3} \)
   D. \( -1 \)
   E. \( -\frac{1}{3} \)

\[ \left( e^{t+1} - t - \frac{1}{(t+1)^2} \right) \]

\[ \left( e^{t+1} - (t+1)^{-2} \right) \]
3. [4 marks]
If \( y = x \ln(x - 2.5) \), then \( y'(3) = \)

A. \( 2 + \ln(0.5) \)
B. \( 3 \ln(0.5) \)
C. \( 3 + \ln(0.5) \)
D. \( 0.5 + \ln(0.5) \)
E. \( 6 + \ln(0.5) \)

\[
\frac{d}{dx} \left( x \ln(x - 2.5) \right) = \ln(x - 2.5) + \frac{x}{x - 2.5}
\]

at \( x = 3 \)

\[
\ln(0.5) + \frac{3}{0.5} = 6 + \ln(0.5)
\]

4. [4 marks]

If \( f(x) = \frac{x^2 + 4e(x^2)}{e^{\sqrt{x}}} \), then \( f'(4) \) equals

A. \( \frac{4 + 3e^{16}}{e^2} \)
B. \( \frac{4 - 3e^{16}}{e^2} \)
C. \( \frac{4 + 3e^{16}}{e^2} \)
D. \( \frac{-8 + 4e^{16}}{e^2} \)
E. \( \frac{-8 + 28e^{16}}{e^2} \)

\[
f'(x) = e^{\sqrt{x}} \left( 2x + 4e^x \cdot \frac{x^2}{2 \sqrt{x}} \right) - \left( \frac{x^2 + 4e^x}{e^{\sqrt{x}}} \right) \frac{e^{\sqrt{x}}}{2 \sqrt{x}}
\]

\[
f'(4) = e^{2} \left( 8 + 32e^{16} \right) - \frac{16 + 4e^{16}}{4} e^2
\]

\[
= \frac{\left( e^{x} \right)^2}{e^2}
\]

\[
= \frac{\left( e^{x} \right)^2}{e^2}
\]

\[
= \frac{8 + 32e^{16} - 4 - e^{16}}{e^2}
\]

\[
= \frac{4 + 3e^{16}}{e^2}
\]

\(\boxed{C}\)
5. [4 marks]

The relationship between national consumption, $C$, and national income, $I$, is given by

$$C^2 + CI - I^2 + 400 = 0$$

where $C$ and $I$ are in billions of dollars. What is the marginal propensity to save when $I = 100$ and $C = 60$ billion dollars?

A. $0.2$

$$2C \frac{dC}{dI} + C + I \frac{dC}{dI} - 2I = 0$$

B. $0.36$

$$120 \frac{dC}{dI} + 60 + 100 \frac{dC}{dI} - 200 = 0$$

C. $0.64$

$$220 \frac{dC}{dI} = 140$$

$$\frac{dC}{dI} = \frac{7}{11}$$

$$\frac{dS}{dI} = 1 - \frac{dC}{dI} = \frac{4}{11} \approx \boxed{0.36}$$

D. $0.8$

E. $1.57$

6. [4 marks]

The relative rate of change of $y$ when $y = (x + 1)^{x-1}$ and $x = 2$ is

A. $0.11$

We are looking for

$$\frac{1}{y} \frac{dy}{dx} \bigg|_{x=2}$$

B. $5.39$

$$\ln y = (x-1) \ln (x+1)$$

C. $9$

$$\frac{1}{x} \frac{dy}{dx} = \frac{x^{x-1}}{x+1} + 2x \ln (x+1)$$

D. $36$

E. $145.65$

$$\bigg|_{x=2} = 1 + 4 \ln 3 \approx \boxed{5.39}$$
7. [4 marks]
If \( R(x) = f(x)g[h(x)] \) and \( f(2) = -2, \ g(2) = -1, \ h(2) = 2, \ f'(2) = 3, \ g'(2) = -3, \ h'(2) = -2 \), then \( R'(2) = \)

A. 5 \[ R'(x) = f'(x)g[h(x)] + f(x)g'(h(x))h'(x) \]
B. 18 \[ R'(2) = f'(2)g(h(2)) + f(2)g'(h(2))h'(2) \]
C. 3 \[ = 3g(2) + (-2)g'(2)h'(2) \]
D. -15 \[ = 3g(2) - 4g'(2) \]
E. -9 \[ = -3 - 12 = -15 \]  \( \boxed{\text{D}} \)

8. [4 marks]
\[
\lim_{{x \to \infty}} \frac{5x^4 - 3x^3 + 7x^2 + 2x - 3}{4x^4 + 5x^3 - 6x^2 + x - 7} = \frac{5}{4} \]  \( \boxed{\text{E}} \)

A. does not exist
B. \( \frac{3}{7} \)
C. \( 5(4!) \)
D. \( \frac{5}{4(4!)} \)
E. \( \frac{5}{4} \)
9. [4 marks]

\[ \lim_{x \to 2} \frac{2^x - 4}{3^x - 9} = \frac{0}{0} \]

So by L'Hopital's rule

\[ \lim_{x \to 2} \frac{2^x \ln 2}{3^x \ln 3} = \frac{4 \ln 2}{9 \ln 3} \]

\[ \text{A. } \frac{4 \ln 2}{9 \ln 3} \]
\[ \text{B. } \frac{4}{9} \]
\[ \text{C. } \frac{\ln 2}{\ln 3} \]
\[ \text{D. } \frac{9 \ln 3}{4 \ln 2} \]
\[ \text{E. does not exist} \]

10. [4 marks]

If two iterations of Newton's method are used to estimate a root of \( f(x) = x^3 + x + 1 \), beginning with the initial estimate \( x_1 = 0 \), then the third estimate, \( x_3 \), is closest to

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ x_{n+1} = x_n - \frac{x_n^3 + x_n + 1}{3x_n^2 + 1} \]

\[ \text{A. } -0.686 \]
\[ \text{B. } -1.000 \]
\[ \text{C. } -0.682 \]
\[ \text{D. } -0.750 \]
\[ \text{E. } -0.720 \]

\[ x_0 = 0 \]
\[ x_1 = 0 - \frac{1}{1} = -1 \]
\[ x_2 = -1 - \frac{(-1 - 1 + 1)}{3 + 1} = -1 - \frac{3}{4} \]

\[ \boxed{-\frac{7}{4}} \]
PART B. Written-Answer Questions

1. [14 marks]
   Let \( f(x) = xe^x \).
   
   (a) Compute \( f'(x) \).
   \[
   f'(x) = e^x + xe^x = (1 + x)e^x
   \]

   (b) Compute \( f''(x) \).
   \[
   f''(x) = e^x + e^x + xe^x = (2 + x)e^x
   \]

   (c) Find the equation of the tangent line to \( y = f'(x) \) at \( x = 1 \). [Note: the tangent line to \( y = f'(x) \), not to \( y = f(x) \)]

   At \( x = 1 \)
   \[
   y = f'(1) = 2e
   \]
   and the slope is \( f''(1) = 3e \)
   
   So the equation of the line is
   \[
   y - 2e = 3e(x - 1)
   \]
   or
   \[
   y = 3ex - e
   \]
2. \(15\) marks

If \(y^2 + xy = x^2 + 1\):

(a) Find \(dx\) in terms of \(x\) and \(y\).

\[ \frac{dx}{dy} = \frac{3x^2 - y}{3y^2 + x} \]

We need this in part (c).

(b) Find \(\frac{dy}{dx}\) when \(x = 1\) and \(y = 1\).

\[ \frac{dy}{dx} = \frac{\frac{3x^2 - y}{3y^2 + x}}{1} \]

2 ways.

A: \(\frac{\partial y}{\partial x} = \left(3x^2 - y\right) \div \left(3y^2 + x\right) \div y \]

Note that when \(x = 1, y = 1\):

\[ \frac{dy}{dx} = \frac{3(1)^2 - 1}{3(1)^2 + 1} = \frac{2}{4} = \frac{1}{2} \]

B: Starting from the first line in (a):

\[ y = \frac{3x^2 - y}{3y^2 + x} \]

\( y = \frac{3x^2 - y}{3y^2 + x} \)

\[ = \frac{3y^2 - y}{3y^2 + x} \]

\( \frac{dy}{dx} = \frac{\frac{3x^2 - y}{3y^2 + x}}{y} \]

\( \frac{dy}{dx} = \frac{\frac{3x^2 - y}{3y^2 + x}}{1} \)

\( \frac{dy}{dx} = \frac{3x^2 - y}{3y^2 + x} \)
3. [15 marks]

The demand equation for a certain product is \( q = 400 - p^2 \).

(a) Find the percentage rate of change of revenue with respect to \( p \) when \( p \) is 10.

\[
\frac{dR}{dp} = \frac{1}{p(400-p^2)} \cdot (400 - 3p^2) = \frac{1}{10 \cdot 300} \cdot 100 = \frac{1}{30} \\
\approx 0.033 \text{ or } 3.33\%
\]

(b) Find the revenue function and the marginal revenue function in terms of \( q \).

\[
p = 400 - q \quad r = pq \quad \frac{dr}{dq} = \frac{800 - 3q}{2(400 - q)}
\]

(c) If \( q = 35m - m^2 \) where \( m \) is the number of employees required to produce \( q \) units of product, find the marginal revenue product when the number of employees is 20.

\[
\frac{dr}{dm} = \frac{dR}{dq} \cdot \frac{dq}{dm} = \frac{800 - 3q}{2(400 - q)} (35 - 2m)
\]

If \( m = 20 \), \( q = 300 \)

\[
so \quad \frac{dr}{dm} = -\frac{100}{2 \cdot \sqrt{100}} (-5) = 25
\]
4. [16 marks]

The demand function \( p(q) \) for a certain product satisfies \( q^2 + 2pq + 4p^2 = 108 \) where \( p > 0 \) and \( q > 0 \) respectively denote unit price and quantity sold.

\[ \frac{dp}{dq} \text{ in terms of } p \text{ and } q. \]

\[ 2q + 2p + 2q \frac{dp}{dq} + 8p \frac{dp}{dq} = 0 \]

\[ \frac{dp}{dq} = - \frac{p + q}{4p + q} \]

\[ \text{(b) Find the point elasticity of demand in terms of } p \text{ and } q. \]

\[ \eta = \frac{\frac{dq}{dp}}{\frac{dp}{dq}} = - \frac{p + q}{4p + q} \]

\[ \eta = -\frac{p}{4p + q} \]

\[ \eta = -1 \]

\[ \frac{p}{q} = 1 \text{ so } 4p + q = p + q \]

\[ 4p = q \]

\[ 2p = q \text{ (-2p = q makes no sense since } p, q > 0) \]

\[ \text{Going back to } \]

\[ q^2 + 2pq + 4p^2 = 108 \]

\[ 4p^2 + 4q^2 + 4q^2 = 108 \]

\[ q^2 + 2q^2 = 108 \]

\[ q = 6 \]

\[ \text{So } p = 3 \text{ and } q = 6 \]