Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a multiple-choice answer sheet, on which you should fill in your name, number, tutorial time, tutorial room, and tutor’s name.

This test consists of 10 multiple choice questions, and 4 written-answer questions. For the multiple choice questions you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter on the answer sheet with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the written-answer questions, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ____________________________________________

GIVEN NAME: ______________________________________________

STUDENT NO: ______________________________________________

SIGNATURE: ________________________________________________

TUTORIAL TIME and ROOM: ___________________________________

REGCODE and TIMECODE: ____________________________________

T.A.'S NAME: _______________________________________________
PART A. Multiple Choice

1. [4 marks]
A professor gives $100,000 to a university to provide for a yearly scholarship forever. If the money can be invested at 5% interest indefinitely, then the scholarship will be:

A. $2,000,000/year
B. $50,000/year
C. $20,000/year
D. $5,000/year
E. $100,000/year

2. [4 marks]
Which nominal annual interest rate compounded continuously is most nearly equivalent to (corresponds to the same effective annual rate as) 6% compounded semiannually?

A. 6.0%
B. 5.9%
C. 5.8%
D. 6.9%
E. 5.7%
3. \( \frac{4}{4} \) marks

A person makes equal annual deposits so that his account will have \$100,000 as soon as he makes the 8\(^{th}\) deposit. If the account earns 4% annually, the amount of each deposit (to the nearest \$100) is:

A. \$10,500
B. \$11,300
C. \$11,100
D. \$10,900
E. \$10,700

\[
R = \frac{100,000}{\frac{8}{0.04}} = \frac{100,000 \times 0.04}{(1.04)^8 - 1}
\]

\[
\approx 10,852.78
\]

$10,900

4. \( \frac{4}{4} \) marks

A couple can afford loan payments of \$2000 per month and they wish to pay off their loan in 15 years. If the interest rate on the loan is 4% compounded monthly then the size of the loan they can afford is closest to:

A. \$277,854
B. \$492,181
C. \$360,000
D. \$270,384
E. \$289,611

\[
A = 2000 \frac{\frac{0.04}{12}}{1 - (1 + \frac{0.04}{12})^{-180}}
\]

\[
\approx 270,384.30
\]

$270,384
5. [4 marks]
If a $6000 loan with annual interest 7% (compounded annually) is to be repaid with payments of $X one year from now, $2000 two years from now, and $3X three years from now, then $X = \frac{6000}{(1.07)^{-1}} + \frac{2000(1.07)^{-2}}{(1.07)^{-1}} + \frac{3X(1.07)^{-3}}{(1.07)^{-1}}$

A. $1302.88
B. $1190.09
C. $1211.35
D. $1338.26
E. $1257.03

\[ X \approx 1257.03 \]

6. [4 marks]
A bond with $10,000 face value, an annual coupon rate of 7.6% with semi-annual coupons, an annual yield of 3.4%, and 6 years to maturity, will sell for (to the nearest $):

\[ P = 10,000(1.017)^{-12} + 380A_{12\,1.017} \]

A. $12,262
B. $10,389
C. $12,245
D. $10,000
E. $9,437

\[ \approx 12,242.30 \]

\[ \# 12,262 \]
7. [4 marks]
A 10-year $200,000 mortgage with weekly payments of $487.60 has an interest rate of 5\% compounded semi-annually. How much principal is repaid in the 1st payment? (You may assume one year is exactly 52 weeks.)

A. $295.29
\[ (1 + \frac{i}{52})^{52} = (1.025)^{2} \]
\[ i = (1.025)^{\frac{1}{26}} - 1 \]
B. $190.03
Interest in 1st payment = 200,000 \times \frac{i}{52} \approx 190.03
C. $384.61
\[ \text{Principal} = R - 190.03 \]
D. $297.57
\[ R \text{ satisfies} \]
\[ 200,000 = R \frac{52d}{i} \]
\[ R = \frac{200,000 i}{1 - (1+i)^{-520}} = \frac{190.03}{1 - (1.025)^{-20}} \approx 487.60 \]
\[ \text{Principal in 1st payment} \times 487.60 - 190.03 = \boxed{297.57} \]
E. $279.27

8. [4 marks]
If A is a 2 × 3 matrix, B is a 3 × 2 matrix, and C is a 3 × 4 matrix, then \((A^T + 2B)^TC\) is

A. a 2 × 3 matrix
\[ \begin{pmatrix} 2 & 2 \\ 3 & A^T + 2B \end{pmatrix} \text{ is } 3 \times 2 \]
B. a 3 × 2 matrix
\[ \begin{pmatrix} 3 & A^T \end{pmatrix} \text{ is } 3 \times 4 \]
C. a 2 × 4 matrix
\[ \begin{pmatrix} 2 \end{pmatrix} \text{ is } 2 \times 4 \]
D. a 3 × 4 matrix
\[ \begin{pmatrix} 3 & C \end{pmatrix} \text{ is } 2 \times 4 \]
E. not defined
9. (4 marks)
Which of the following is a solution to \( AX = B \), where

\[
A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix}
\]

\[
X = A^{-1}B
\]

A. \( X = \frac{1}{3} \begin{pmatrix} 17 & -10 \\ 3 & 12 \end{pmatrix} \)

B. \( X = \frac{1}{6} \begin{pmatrix} 13 & 10 \\ 5 & 3 \end{pmatrix} \)

C. \( X = \frac{1}{3} \begin{pmatrix} 20 & 10 \\ 16 & 8 \end{pmatrix} \)

D. \( X = \frac{1}{3} \begin{pmatrix} 15 & 3 \\ 3 & -6 \end{pmatrix} \)

E. \( X = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \)

\[
A^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}
\]

\[
X = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 20 & 10 \\ 16 & 8 \end{pmatrix}
\]

10. (4 marks)
The system

\[
\begin{cases} 
x + 2y + 7z = 6 \\
2y + 4z = 1 \\
x + 2y + (b+1)z = 8
\end{cases}
\]

has no solution if \( b \) is:

A. 5
B. 6
C. 7
D. 8
E. 9

No solution only if \( b = 6 \)
PART B. Written-Answer Questions

1. [15 marks]
A $500,000 mortgage is amortized over 25 years with monthly payments.

5. (a) If the interest rate is 4\% compounded semi-annually then find the size of the monthly payment.

\[(1 + \frac{i}{2})^{2} = (1.02)^{2}\]
\[i = 0.0330589\]
\[500,000 = R \times a_{\frac{25}{2}}\]
\[R = \frac{500,000}{a_{\frac{25}{2}}i} = \frac{500,000 \times \frac{i}{2}}{1 - (1 + \frac{i}{2})^{-250}} = \frac{500,000 \times \frac{0.0330589}{2}}{1 - (1.0152)^{-250}} \approx \$2630.10\]

5. (b) Find the outstanding principal at the end of 5 years.

60 payments made, 240 remain.

\[P(t) = R \times a_{\frac{240}{12}} = 2630.10 \times \left[1 - (1.0152)^{-40}\right] \approx \$435,269.41\]

5. (c) In 5 years the mortgage must be renewed. At that time the interest rate is 3\% compounded monthly. If the remaining principal is amortized over the remaining 20 years, then find the new monthly payment.

Now \[(1 + \frac{i}{12})^{12} = (1.015)^{12}\]
\[i = 0.02484516\]
\[435,269.41 = R \times a_{\frac{240}{12}}\]
\[R = \frac{435,269.41 \times \frac{i}{12}}{1 - (1 + \frac{i}{12})^{-240}} = \frac{435,269.41 \times \left[(1.015)^{12} - 1\right]}{\left[1 - (1.015)^{-40}\right]} \approx \$2409.95\]

Because of the phrasing of the question, \[i = \frac{0.3}{12} = 0.0025\]
and \[R = \frac{435,269.41 \times 0.0025}{1 - (1.0025)^{-240}} \approx \$2414\]

was also acceptable.
2. [15 marks] (Long question)
(a) Find the inverse of

\[
A = \begin{pmatrix}
1 & -1 & 3 \\
3 & 3 & 0 \\
0 & 2 & 1
\end{pmatrix}
\]

(Don’t be afraid if there are lots of fractions in your answer.)

\[
\begin{pmatrix}
1 & -1 & 3 \\
3 & 3 & 0 \\
0 & 2 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & 3 \\
0 & 6 & -9 \\
0 & 2 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & 3 \\
0 & 1 & -\frac{3}{2} \\
0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & 3 \\
0 & 1 & -\frac{3}{2} \\
0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & 3 \\
0 & 1 & -\frac{3}{2} \\
0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -\frac{3}{2} \\
0 & 0 & 1
\end{pmatrix}
\]

\[
A^{-1} = \begin{pmatrix}
\frac{1}{8} & \frac{7}{24} & -\frac{3}{8} \\
-\frac{1}{8} & \frac{1}{24} & \frac{3}{8} \\
\frac{1}{4} & -\frac{1}{12} & \frac{1}{4}
\end{pmatrix}
\]

(b) Using your answer in (a), solve the system

\[
\begin{cases}
x - y + 3z = 1 \\
3x + 3y = 2 \\
2y + z = -1
\end{cases}
\]

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= A^{-1} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}
= \begin{pmatrix}
\frac{7}{24} & -\frac{3}{8} \\
-\frac{1}{8} & \frac{3}{8} \\
\frac{1}{4} & -\frac{1}{12} & \frac{1}{4}
\end{pmatrix}
\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}
= \begin{pmatrix}
\frac{26}{24} \\
-10/24 \\
-2/12
\end{pmatrix}
\]

\[
x = \frac{13}{12} \\
y = -\frac{5}{12} \\
z = -\frac{1}{12}
\]
3. (15 marks)
A fund manager is setting up a $150 million fund with stocks from hi-tech, energy, and media sectors. The manager wants twice as much (in dollars) hi-tech as media stock, and three times as much (in dollars) energy as hi-tech and media combined. How much (in dollars) should he/she buy from each sector?

Let $H =$ amount of hi-tech
$E =$ " " energy
$M = ""$ media

\[
\begin{align*}
H + E + M &= 150 \text{ million} \\
H &= 2M \\
E &= 3(H + M)
\end{align*}
\]

Easiest way: $H = 2M$
$E = 3(2M + M) = 9M$

$2M + 9M + M = 150,000,000$
$12M = 150,000,000$

$M = 12,500,000$
$H = 25,000,000$
$E = 112,500,000$

\[
\begin{bmatrix}
H \\
E \\
M
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 2 \\
3 & -1 & 3
\end{bmatrix}\begin{bmatrix}
150 \\
0 \\
0
\end{bmatrix}
\]

$R_1 \rightarrow R_1 - R_2$
$R_3 \rightarrow R_3 - 3R_1$

\[
\begin{bmatrix}
1 & 1 & 1 & 150 \\
0 & -1 & -3 & -150 \\
0 & -4 & 0 & 450
\end{bmatrix} 
\rightarrow 
\begin{bmatrix}
1 & 1 & 1 & 150 \\
0 & 1 & 3 & 150 \\
0 & 0 & 12 & 150
\end{bmatrix}
\]

So
\[
\begin{align*}
H &= 25 \text{ million} \\
E &= 112.5 \text{ million} \\
M &= 12.5 \text{ million}
\end{align*}
\]
4. [15 marks]

A debt of $50,000 is amortized at a nominal rate of 6% over 5 years with monthly payments, the first payment being at the end of the first month.

\[ R = \frac{50,000 \times 0.005}{1 - (1.005)^{-60}} \approx \$966.64 \]

Just after the tenth payment, the debtor is able to make an extra $5000 payment toward the debt. If the debtor continues to make regular payments, except for the last one which is smaller,

(b) How many payments are required altogether to pay off the debt?

\[ n = \frac{\ln \left( \frac{7659}{966.64} \right)}{\ln (1.005)} \approx 53.45 \]

It takes \textbf{54 payments in all}.

(c) How much is the last smaller payment?

\[ 50,000 \times (1.005)^{54} - 5000 \times (1.005)^{44} \leq \frac{531.005 \times (1.005)^{44}}{966.64} \times X \]

Subtracting: \[ X \approx \$439.35 \]

Note that \( 0.45 \times 966.64 \approx \$435 \). Even when unrounded \( n = 45.390... \) gets \$438.76. Closer, but not quite right.