Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 12 pages. In addition, you should have a multiple-choice answer sheet, on which you should fill in your name, number, tutorial time, tutorial room, and tutor’s name.

This test consists of 10 multiple choice questions, and 4 written-answer questions. For the multiple choice questions you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter on the answer sheet with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the written-answer questions, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _________________________________________________

GIVEN NAME: ________________________________________________

STUDENT NO: _________________________________________________

SIGNATURE: __________________________________________________

TUTOR CODE and TIMECODE: ___________________________________

T.A.’S NAME: __________________________________________________

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FOR MARKER ONLY

Multiple Choice

- B1
- B2
- B3
- B4

TOTAL
PART A. Multiple Choice

1. [4 marks]
Let \( M \) be the absolute maximum of
\[
h(x) = 2x^3 - 3x^2 - 12x + 27
\]
on \([-3, 3]\) and let \( m \) be the absolute minimum of the same function on the same interval. Then \( m + M = \)

A. \( 41 \)
B. \( 25 \)
C. \( -11 \)
D. \( 0 \)
E. \( 16 \)

\[ h'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1) \]
Critical points at \( x = -1 \) and \( x = 2. \)
\[ h(-3) = -18 = m \]
\[ h(-1) = 34 = M \]
\[ m + M = 16 \]

2. [4 marks]
Assume \( k \) is some constant. The function
\[
g(x) = x^4 - 6kx^2 + 6k^2x
\]
has at least one point of inflection

A. if and only if \( k > 0 \)
B. if and only if \( k \geq 0 \)
C. for any \( k \)
D. if and only if \( k < 0 \)
E. for no value of \( k \)

\[ g'(x) = 4x^3 - 12kx + 6k^2 \]
\[ g''(x) = 12x^2 - 12k = 12(x^2 - k) \]
If \( k < 0 \), \( g''(x) > 0 \), hence never changes sign, so no points of inflection.
If \( k > 0 \), \( g''(x) = 12(x - \sqrt{k})(x + \sqrt{k}) \), changes signs (twice actually) at least once, hence has at least one point of inflection.
3. \[4 \text{ marks}\]
If the weekly sales volume \( q \) for a product is given by \( q = \frac{32,000}{(p + 8)^{3/5}} \), where \( p \) is the price in dollars, then the approximation found by using differentials for the change in sales if the price is raised from \$24.00 to \$24.50 is

A. \(-50\)
B. \(-49\)
C. \(-800\)
D. \(-51\)
E. \(50\)

\[
dq = \frac{2 \times 32,000 \times \frac{1}{2}}{(24.5)^{3/5}} \]
\[
= \frac{1}{2} \times 32,000 \]
\[
= \frac{1}{2} \times 1000 = -50 \quad (A)
\]

4. \[4 \text{ marks}\]
If \( x_1 = 2 \) is used as a first estimate for a root of the equation \( e^x = x^3 \), then to 2 decimal places, Newton’s method gives the second estimate \( x_2 = \)

A. 1.87
B. 1.85
C. 1.86
D. 1.88
E. 1.89

\[
f(x) = e^x - x^3 \quad \text{or} \quad x^3 - e^x
\]

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

\[
x_{n+1} = x_n - \frac{e^{x_n} - x_n^3}{e^{x_n} - 3x_n^2}
\]

If \( x_1 = 2 \),

\[
x_2 = 2 - \frac{e^2 - 8}{e^2 - 12} \approx 1.8675 \quad (A)
\]
5. \( \lim_{x \to 0^+} \frac{1}{z} \ln\left(\frac{1+x}{1-x}\right) = \lim_{x \to 0^+} \frac{\ln(1+x) - \ln(1-x)}{x} \quad \frac{0}{0} \text{ type} \)

A. \( = 0 \)
B. \( = 1 \)
C. \( = e^2 \)
D. \( = 2 \)
E. \( \text{is undefined} \)

More complicated version:

\[ \lim_{x \to 0^+} \frac{\ln\left(\frac{1+x}{1-x}\right)}{x} \quad \frac{0}{0} \text{ type} \]

\[ = \lim_{x \to 0^+} \frac{1 - x}{1 + x} \cdot \frac{[(1-x) - (1+x)(1-x)]}{(1-x)^2} = \lim_{x \to 0^+} \frac{2}{1+x(1-x)} \]

\[ = 2 \quad \boxed{D} \]

6. \( \lim_{x \to 1} \frac{1}{\sqrt{x} - 1} \)

A. \( 2 \)
B. \( e^2 \)
C. \( e^{-\frac{1}{2}} \)
D. \( e^2 \)
E. \( \frac{1}{2} \)

Let \( y = \frac{1}{\sqrt{x} - 1} \)

\( \lim_{x \to 1} \ln y = \lim_{x \to 1} \frac{\ln x}{\sqrt{x} - 1} \quad \frac{0}{0} \text{ type} \)

\( = \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = 2 \)

So \( \lim_{x \to 1} y = e^2 \quad \boxed{B} \)
7. [4 marks]

If \( F(x) = \int_3^x \ln (t^2 + 1) \, dt \), then \( F'(4) \) is closest to

A. 2.83  \( F'(x) = \ln (x^2 + 1) \)

B. 0.53  \( F'(4) \approx 2.83 \)  \( \text{\checkmark} \)

C. 0.00

D. 0.47

E. -0.13

8. [4 marks]

\[
\int_0^1 \frac{x-1}{x+1} \, dx
\]

A. 1 + 2ln 2

B. -\frac{1}{2}

C. 1 - ln 4

D. -\frac{2}{3}

E. -\frac{1}{2} ln 2

\[
\int_0^1 \frac{x-1}{x+1} \, dx = \int_0^1 \left[ 1 - \frac{2}{x+1} \right] \, dx = 1 - 2 \ln 2
\]

\[
= \left[ x - 2 \ln |x+1| \right]_0^1 = 1 - 2 \ln 2
\]

\[
= 1 - \ln 2^2 = 1 - \ln 4 \quad \text{\( \checkmark \)}
\]
9. \[4 \text{ marks}\]
\[
\int_{1}^{8} \frac{3dx}{x^{\frac{3}{4}}(x^{\frac{1}{4}} + 1)^{2}} = \\
[\text{Hint: Try a substitution.}]
\]
A. \(\frac{3}{2}\)
B. \(96\)
C. \(\frac{63}{8}\)
D. \(-\frac{3}{2}\)
E. \(\frac{1}{6}\)

Let \(u = x^{\frac{3}{4}} + 1\), \(du = \frac{1}{3}x^{\frac{1}{4}}dx\)

\[\int_{1}^{3} \frac{3du}{u^{2}} = \int_{2}^{3} \frac{9du}{u^{2}} = \left[ \frac{9u^{-1}}{-1} \right]_{2}^{3} = \frac{9}{2} \quad \text{when } x = 1\]
\[= \left[ \frac{9}{2} \right]_{2}^{3} = 9 \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{9}{2} \quad \text{A}\]

10. \[4 \text{ marks}\]
The area bounded by the graphs of \(y = x^{2} - 1\) and \(y = 2x + 2\) is

A. \(4\)
B. \(\frac{32}{3}\)
C. \(\frac{34}{3}\)
D. \(\frac{45}{2}\)
E. \(\frac{57}{2}\)

\[
\text{Area} = \int_{-1}^{3} \left[ (2x+2) - (x^{2} - 1) \right] dx \\
= \int_{-1}^{3} (2x^{2} - x + 3) dx = \left[ \frac{x^{3} - \frac{x^{2}}{2} + 3x} {3} \right]_{-1}^{3} \\
= \left( 9 - 9 + 9 \right) - \left( 1 + \frac{1}{2} - 3 \right) = 9 - \left( -\frac{5}{2} \right) = \frac{32}{3} \quad \text{B}\]

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PART B. Written-Answer Questions

1. (15 marks)

Let \( f(x) = \ln|x^3 + 1| \).

You may assume that \( f'(x) = \frac{3x^2}{x^3 + 1} \) and \( f''(x) = \frac{3x(2 - x^3)}{(x^3 + 1)^2} \).

(a) Find the critical points, intervals where the function is increasing or decreasing, local extrema, intervals where the function is concave up or down, inflection points, and asymptotes, if any.

\[
\lim_{x \to \pm \infty} f(x) = \infty \quad \text{so} \quad \text{no H.A.} \quad (\text{since } |x^3 + 1| \to \infty, \quad \ln |x^3 + 1| \to \infty).
\]

\[
\lim_{x \to -1} f(x) = -\infty, \quad \text{\(x = -1\) is a V.A.} \]

\( f'(x) = 0 \) at \( x = 0 \) and fails to exist at \( x = -1 \)

(whence \( f \) fails to exist as well).

\[\begin{array}{c|cccc}
\infty & - & \text{dec} & - \\
-1 & + & \text{inc} & - \\
0 & + & \text{inc} & - \\
\infty & + & \text{inc} & - \\
\end{array}\]

\( f''(x) = 0 \) at \( x = 0 \) and \( x = 2^{\frac{1}{3}} \) and fails to exist at \( x = -1 \)

(as does \( f' \)).

\[
\begin{array}{c|cccc}
\infty & - & \text{conc down} & - \\
-1 & - & \text{conc down} & - \\
0 & + & \text{conc up} & - \\
2^{\frac{1}{3}} & - & \text{conc down} & - \\
\end{array}
\]

Question continues on Page 8
(b) Draw a careful sketch of the graph of \( y = f(x) \), labelling the important features.
2. [10 marks]
A certain tavern sells beer only, no wine or spirits, and has seating for 100 customers. If it charges $3 per beer it will be full but each 10¢ increase in the price of a beer results in one empty seat. Find the price per beer that the tavern should charge to maximize its revenue. You may assume that the time required to drink a beer is the same for all customers and is independent of the price per beer. Show that the price you have found actually maximizes revenue.

Let \( x \) = number of $0.10 \ increases, so

\[
\begin{align*}
\frac{p}{x} &= 3 + 0.10x \\
q &= 100 - x \\
R &= pq = (3 + 0.10x)(100 - x) \\
&= 300 + 7 - 10x \\
\end{align*}
\]

\( 0 \leq x \leq 100 \) a closed bounded interval.

\[
\frac{dR}{dx} = -10(100-x) - (3 + 0.10x) = 7 - 20x
\]

\( \frac{dR}{dx} = 0 \) when \( x = 35 \), so \( P = \$6.50 \)

Arguments for max:

1) \( R'' = -2 < 0 \) for every \( x \)
2) \( R \) is a quadratic with negative coefficient of \( x^2 \) so critical point is absolute max.
3) Since we have a closed, bounded interval

\[
\begin{align*}
x &= 0 \quad R(0) = 300 \\
x &= 35 \quad R(35) = 65 \times 6.5 = 422.50 \quad \text{max} \\
x &= 100 \quad R(100) = 0
\end{align*}
\]
3. \(15\) marks

The demand function for a product is given by \(p = 100e^{-0.1q}\).

(a) For what values of \(q\) is demand elastic?

\[
\eta = \frac{p}{q} \frac{dp}{dq} = \frac{100e^{-0.1q}}{q} \cdot (-10)e^{-0.1q} = \frac{-10}{q}
\]

\[|\eta| = \frac{10}{q} > 1 \quad \text{when} \quad q < 10\]

(b) If the equilibrium price is \(p_0 = 10\), find Consumers' Surplus to two decimal places.

\[
\text{When } p_0 = 10 = 100e^{-1.1q_0} \Rightarrow e^{-1.1q_0} = \frac{1}{10}
\]

\[
\Rightarrow e^{1.1q_0} = 10 \Rightarrow 1.1q_0 = \ln 10 \Rightarrow q_0 = 10 \ln 10 \approx 23.026
\]

\[
CS = \int_0^{q_0} [D(q) - p_0] \, dq
\]

\[
= \int_0^{\ln 10} [100e^{-1.1q} - 10] \, dq
\]

\[
= \left[ \frac{-1000e^{-1.1q} - 10q} {10 \ln 10} \right]_0^{\ln 10}
\]

\[
= (1000 - 100 \ln 10) + 1000
\]

\[
= 2697.64
\]
Evaluate (numerical answers to two decimal places please)

\[ 4. \text{(80 marks)} \]

\[ 6. \text{(a) } \int_0^1 x \sqrt{x+1} \, dx \]

**By substitution:** Let \( u = x + 1 \), \( du = dx \)

\[ \int_1^2 (u^{3/2} - u^{1/2}) \, du = \left[ \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right] \bigg|_1^2 \]

\[ = \left( \frac{2}{5} \cdot 4 \sqrt{2} - \frac{2}{3} \cdot 2 \sqrt{2} \right) - \left( \frac{2}{5} - \frac{2}{3} \right) \approx 0.64 \approx 0.64 \]

**Or:** By parts

Let \( u = x \), \( dv = \sqrt{x+1} \, dx \)

\[ du = dx, \quad v = \frac{2}{3} (x+1)^{3/2} \]

\[ = \frac{4}{3} \sqrt{2} + \frac{4}{3} \cdot \frac{2}{3} \sqrt{2} - \frac{4}{3} \cdot \frac{2}{3} (x+1)^{3/2} \bigg|_0 \]

\[ = \frac{4}{3} \sqrt{2} + \frac{4}{3} \cdot \frac{2}{3} \sqrt{2} - \frac{4}{3} \cdot \frac{2}{3} (x+1)^{3/2} \bigg|_0 \approx 0.64 \approx 0.64 \]

\[ 6. \text{(b) } \int_2^3 \frac{dx}{x(x-1)^2} \]

\[ \frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \]

\[ A(x-1)^2 + Bx(x-1) + Cx = 1 \]

\[ x=1 \Rightarrow C=1; \quad x=0 \Rightarrow A = 1; \quad \text{any other } x, \text{ say } x = 2 \]

\[ A + 2B + 2C = 1 \Rightarrow 1 + 2B + 2 = 1 \Rightarrow 2B = -2 \Rightarrow B = -1 \]

\[ \int_2^3 \frac{dx}{x(x-1)^2} = \int_2^3 \left( \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2} \right) \, dx \]

\[ = \left[ \ln|x| - \ln|x-1| - \frac{1}{x-1} \right]_2^3 = \ln \left( \frac{3}{2} \right) - \ln 2 - \left( \frac{1}{2} - 1 \right) \]

\[ = \ln \left( \frac{3}{2} \right) + \frac{1}{2} \approx 0.21 \]
Find the accumulated value of a continuous annuity after 5 years, if the payment at time $t$ (in years) is at the rate of $1000t$ dollars per year and the annual rate of interest is 10% compounded continuously.

\[ S = \int_0^5 f(t)e^{r(T-t)}\,dt \]

\[ = \int_0^5 1000te^{-10(5-t)}\,dt \]

\[ = 1000e^{5} \int_0^5 te^{-10t}\,dt \quad \text{using } du = e^{-10t}\,dt \]

\[ = 1000e^{5} \left[ -\frac{te^{-10t}}{10} \bigg|_0^5 + \frac{1}{10} \int_0^5 e^{-10t}\,dt \right] \]

\[ = 10,000e^{5} \left[ -\frac{5e^{-5}}{10} - 10e^{-10t} \bigg|_0^5 \right] \]

\[ = 10,000e^{5} \left[ -\frac{5e^{-5}}{10} + 10e^{5} - 10e^{5} \right] \]

\[ = 10,000e^{5} \left[ 10 - 15e^{-5} \right] \]

\[ = 10,000 \left[ 10e^{5} - 15 \right] \]

\[ \approx \$ 14,872.13 \]