FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2014

MAT 133Y1Y
Calculus and Linear Algebra for Commerce

Duration: 3 hours
Examiners: A. Igelfeld
P. Kergin
L. Shorser
J. Tate

FAMILY NAME: ____________________________

GIVEN NAME: ____________________________

STUDENT NO: ____________________________

SIGNATURE: ____________________________

NOTE:

1. Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

2. Instructions: Fill in the information on this page, and make sure your test booklet contains 14 pages.

3. This exam consists of 15 multiple choice questions, and 5 written-answer questions. For the multiple choice questions you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter on the front page with your pencil. Each correct answer is worth 3 marks; a question left blank, or an incorrect answer or two answers for the same question is worth 0. For the written-answer questions, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

4. Put your name and student number on each page of this examination.

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ANSWER BOX FOR PART A
Circle the correct answer

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PART A. MULTIPLE CHOICE

1. [3 marks]

If \( z \) is used as the parameter in the solution set of the system
\[
2x + 3y + 5z = 3 \\
3x + 4y + 6z = 5
\]
then \( x = \)

A. \( 2 - 3z \)  \\
B. \( 3 + 2z \)  \\
C. \( -2 + z \)  \\
D. \( 1 + 3z \)  \\
E. \( -1 - 3z \)

2. [3 marks]

If \( f(x) = \sqrt{3x - 5} \), then \( f'(2) = \)

A. \( 0 \)  \\
B. \( -\frac{1}{2} \)  \\
C. \( 1 \)  \\
D. \( -1 \)  \\
E. \( \frac{1}{2} \)
3. [3 marks]
If \( y(x) \) satisfies \( y^4 + 1 = xy + x^2 \) and \( y = 1 \) when \( x = 1 \), then when \( x = 1 \), \( y' = \)
A. \(-\frac{1}{2}\)
B. \(\frac{1}{4}\)
C. \(-1\)
D. \(\frac{1}{2}\)
E. 1

4. [3 marks]
If \( 2x^y = e^2y \), then when \((x,y) = (e,2)\), \( \frac{dy}{dx} = \)
A. \(\frac{e}{4}\)
B. \(-\frac{e}{2}\)
C. \(-\frac{4}{e}\)
D. \(2e\)
E. \(-\frac{2}{e}\)
5. [3 marks]
On the interval \([-2, 4]\), the function \(f(x) = 2x^3 - 9x^2\) has
A. an absolute minimum at \(x = 3\) and an absolute maximum at \(x = 4\).
B. an absolute minimum at \(x = -2\) and no absolute maximum.
C. an absolute minimum at \(x = 3\) and an absolute maximum at \(x = 0\).
D. an absolute minimum at \(x = -2\) and an absolute maximum at \(x = 0\).
E. an absolute minimum at \(x = -2\) and an absolute maximum at \(x = 4\).

6. [3 marks]
\[
\lim_{x \to 1} \frac{x - 1 - \ln x}{x - 2\sqrt{x} + 1} =
\]
A. \(\frac{1}{2}\)
B. 1
C. 2
D. \(\infty\)
E. 0
7. [3 marks]
If \(x_1 = 0\) is used as a first estimate to approximate a root of \(x^3 + x = 1\) by Newton’s method, then the third estimate, \(x_3\), equals

A. \(\frac{3}{4}\)  
B. \(\frac{2}{3}\)  
C. \(\frac{5}{6}\)  
D. \(\frac{1}{2}\)  
E. \(\frac{5}{8}\)

8. [3 marks]
\[
\int_1^e \left( \frac{1}{x} - \frac{1}{x^2} \right) dx =
\]

A. \(\frac{1}{e} - \frac{1}{e^2}\)  
B. \(-\frac{1}{e^2} + 2\)  
C. \(\frac{1}{e} - 1\)  
D. \(\frac{1}{e}\)  
E. 1
9. [3 marks]
\[ \int_0^2 x^2 \sqrt{1 + x^3} \, dx = \]
A. \( \sqrt{35} \)
B. \( \sqrt{6} \)
C. \( \frac{26}{3} \)
D. \( \frac{52}{9} \)
E. 36

10. [3 marks]
Let \( g(x) = \int_1^x \sqrt{2^t + 1} \, dt \). Then \( g'(2) = \)
A. \( \sqrt{5} \)
B. \( \sqrt{5} - \sqrt{2} \)
C. \( \frac{2 \ln 2}{\sqrt{5}} \)
D. \( \sqrt{2^2 + 1} \)
E. 0
11. [3 marks]

Let \( f(x, y) = x^2 y^3 + e^{xy} \). Then \( f_y = \)

A. \( 6xy^2 + yxe^{xy} \)
B. \( 3y^2 + e^x \)
C. \( 3x^2 y^2 + e^{xy} \)
D. \( 3x^2 y^2 + ye^{xy} \)
E. \( 3x^2 y^2 + xe^{xy} \)

12. [3 marks]

If \( y + xz^3 - x^2 z^3 = 2 \) defines \( z \) implicitly as a function of \( x \) and \( y \) near the point \( x = 1, y = 2, z = 1 \), then at that point \( \frac{\partial z}{\partial y} = \)

A. \(-1\)
B. \(1\)
C. \(0\)
D. \(-6\)
E. \(5\)
13. [3 marks]  
If $f(x, y) = x^2e^{y^2}$ then $f_{xy}(1, 1) =$  
A. $4e$  
B. $12e$  
C. $6e$  
D. $8e$  
E. $2e$

14. [3 marks]  
If $z = x^2 + xy + y^2$ where $x = 3t - 6$ and $y = t^2 + 2$ then when $t = 1$ \[ \frac{dz}{dt} = \]  
A. 9  
B. 3  
C. -3  
D. 15  
E. 6
15. [3 marks]
The joint demand functions for the products $A$ and $B$ are given by:

$$q_A = \frac{200}{p_A \sqrt{p_B}}$$

$$q_B = \frac{300}{p_B \sqrt[p]{p_A}}$$

Which of the following statements is true?

A. $\frac{\partial q_A}{\partial p_A} > 0$
B. $\frac{\partial q_B}{\partial p_B} > 0$
C. Products $A$ and $B$ are complementary
D. Products $A$ and $B$ are competitive
E. Products $A$ and $B$ are neither complementary nor competitive
PART B. WRITTEN-ANSWER QUESTIONS

B1. [11 marks]

(a) [4 marks]
A 20 year mortgage for $500,000 has monthly payments with interest at 4% compounded semiannually. Find the amount of each payment (to the nearest cent).

(b) [3 marks]
Find the principal outstanding (to the nearest cent) in the mortgage of question 1.(a), just after the 144th payment has been made.

(c) [4 marks]
What is the market price (to the nearest cent) of a $100 bond having 9 years until maturity and semiannual coupons, with annual coupon rate 6% and annual yield rate 5%?
B2. [11 marks]
Consider the graphs of $x = y^2 + 1$ and $x = 4y + 1$

(a) [2 marks]
Find the points where those graphs intersect.

(b) [5 marks]
Express as an integral the finite area bounded by those graphs.

(c) [4 marks]
Find the area from part (b).
B3. \[13\] marks
Evaluate the following integrals

(a) \[6\] marks
\[
\int_{0}^{1} (2x + 1)e^{2x} \, dx
\]

(b) \[7\] marks
\[
\int_{1}^{\infty} \frac{1}{x^2(x+1)} \, dx
\]
B4. [8 marks]

Assuming that $y > 0$, find an expression for $y$ in terms of $x$ if $y$ satisfies the differential equation

$$\frac{dy}{dx} = xy$$

and $y = 3$ when $x = 0$. [Hint: This one’s easy.]
B5. [12 marks]
(a) [6 marks]
Find and classify the critical point(s) of

\[ f(x, y) = 5x^2 - 2xy + 2y^2 - 10x + 2y \]

(b) [6 marks]
By using the method of Lagrange multipliers only find the critical points of the joint cost function

\[ c(q_A, q_B) = q_A^2 - q_A q_B + \frac{3}{2} q_B^2 + 300 \]
subject to the constraint

\[ q_A + q_B = 700. \]

[Show all your work. No marks will be given for any other method.]
PART A. MULTIPLE CHOICE

1. [3 marks]
   If \( z \) is used as the parameter in the solution set of the system
   \[
   \begin{align*}
   2x + 3y + 5z &= 3 \\
   3x + 4y + 6z &= 5
   \end{align*}
   \]
   then \( x = \)
   A. \( 2 - 3z \) \\
   B. \( 3 + 2z \) \\
   C. \( -2 + z \) \\
   D. \( 1 + 3z \) \\
   E. \( -1 - 3z \)

2. [3 marks]
   If \( f(x) = \frac{\sqrt{3x - 5}}{x} \), then \( f'(2) = \)
   A. \( 0 \) \\
   B. \( -\frac{1}{2} \) \\
   C. \( 1 \) \\
   D. \( -1 \) \\
   E. \( \frac{1}{2} \)

   Quotient Rule: \( f'(x) = \frac{x \cdot \frac{1}{2\sqrt{3x - 5}} - 3 - \sqrt{3x - 5}}{x^2} \)
   \[
   f'(2) = \frac{\frac{2\sqrt{3}}{4} - \sqrt{1}}{4} = \frac{1}{2}, \quad \text{E}
   \]
3. [3 marks]
If \( y(x) \) satisfies \( y^4 + 1 = xy + x^2 \) and \( y = 1 \) when \( x = 1 \), then when \( x = 1 \), \( y' = \)

A. \(-\frac{1}{2}\)
B. \(\frac{1}{4}\)
C. \(-1\)
D. \(\frac{1}{2}\)
E. 1

\[ 4y^3y' = y + xy' + 2x. \] Substituting \( x = 1, y = 1, \)
\[ 4y' = 1 + y' + 2 \]
\[ 3y' = 3 \]
\[ y' = 1, \quad E \]

4. [3 marks]
If \( 2x^y = e^2y \), then when \((x, y) = (e, 2)\), \( \frac{dy}{dx} = \)

A. \(\frac{e}{4}\)
B. \(-\frac{e}{2}\)
C. \(-\frac{4}{e}\)
D. \(2e\)
E. \(-\frac{2}{e}\)

Taking ln of both sides:

\[ \ln 2 + y \ln x = 2 + \ln y \]
\[ y' \ln x + \frac{y}{x} = \frac{1}{y}y' \]
\[ y' \ln e + \frac{2}{e} = \frac{1}{2}y' \]
\[ y' - \frac{1}{2}y' = -\frac{2}{e} \]
\[ \frac{1}{2}y' = -\frac{2}{e} \]
\[ y' = -\frac{4}{e}, \quad C \]
5. [3 marks]
On the interval \([-2, 4]\), the function \(f(x) = 2x^3 - 9x^2\) has

A. an absolute minimum at \(x = 3\) and an absolute maximum at \(x = 4\).
B. an absolute minimum at \(x = -2\) and no absolute maximum.
C. an absolute minimum at \(x = 3\) and an absolute maximum at \(x = 0\).
D. an absolute minimum at \(x = -2\) and an absolute maximum at \(x = 0\).
E. an absolute minimum at \(x = -2\) and an absolute maximum at \(x = 4\).

\(f'(x) = 6x^2 - 18x = 6x(x - 3)\)
Critical points are \(x = 0\) and \(x = 3\)
Endpoints of closed intervals are \(x = -2\) and \(x = 4\)
The continuous function \(f\) must have absolute max and min among these 4 points.
\(f(-2) = 2(-8) - 36 = -52\) \(\text{min}\) \(x = -2\)
\(f(0) = 0\) \(\text{max}\) \(x = 0\)
\(f(3) = 54 - 9 \cdot 9 = -27\)
\(f(4) = 2 \cdot 64 - 9 \cdot 16 = -16,\) \(D\)

6. [3 marks]
\[\lim_{x \to 1} \frac{x - 1 - \ln x}{x - 2\sqrt{x} + 1}\]

A. \(\frac{1}{2}\)
B. 1
C. 2
D. \(\infty\)
E. 0

\[\lim_{x \to 1} \frac{x - 1 - \ln x}{x - 2\sqrt{x} + 1} = \frac{0}{0}\]
\[= \lim_{x \to 1} \frac{1 - \frac{1}{x}}{1 - \frac{1}{\sqrt{x}}} \text{ still } \frac{0}{0}\]
\[= \lim_{x \to 1} \frac{\sqrt{x}}{2x^{3/2}} = \frac{1}{2} = 2,\] \(C\)
7. [3 marks]
If $x_1 = 0$ is used as a first estimate to approximate a root of $x^3 + x = 1$ by Newton’s method, then the third estimate, $x_3$, equals
A. $\frac{3}{4}$
B. $\frac{2}{3}$
C. $\frac{5}{6}$
D. $\frac{1}{2}$
E. $\frac{5}{8}$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

where $f(x) = x^3 + x - 1$

$x_1 = 0$

$x_2 = 0 - \frac{-1}{1} = 1$

$x_3 = 1 - \frac{1 + 1 - 1}{4} = 1 - \frac{1}{4} = \frac{3}{4}$. \textbf{A}

8. [3 marks]

$$
\int_{1}^{e} \left( \frac{1}{x} - \frac{1}{x^2} \right) \, dx =
$$

A. $\frac{1}{e} - \frac{1}{e^2}$
B. $\frac{1}{e^2} + \frac{2}{e^3}$
C. $\frac{1}{e} - 1$
D. $\frac{1}{e}$
E. 1

$$
\left[ \ln|x| + \frac{1}{x} \right]_{1}^{e} = (\ln e + \frac{1}{e}) - (\ln 1 + 1)
= 1 + \frac{1}{e} - 1 = \frac{1}{e}. \quad \textbf{D}
$$
9. [3 marks]
\[\int_0^2 x^3 \sqrt{1 + x^3} \, dx = \]
A. \(\sqrt{35}\)
B. \(\sqrt{5}\)
C. \(\frac{26}{3}\)
D. \(\frac{52}{9}\)
E. 36

Let \(u = 1 + x^3\) \quad x = 0 \quad u = 1
\[\frac{du}{dx} = 3x^2 \quad x = 2 \quad u = 9\]
\[\frac{1}{3} \int_1^9 \sqrt{u} \, du = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} \bigg|_1^9\]
\[= \frac{2}{9} (27 - 1)\]
\[= \frac{52}{9}, \quad D\]

10. [3 marks]
Let \(g(x) = \int_1^x \sqrt{2t + 1} \, dt\). Then \(g'(2) = \)
A. \(\sqrt{5}\)
B. \(\sqrt{5} - \sqrt{2}\)
C. \(\frac{2 \ln 2}{\sqrt{5}}\)
D. \(\sqrt{2^x + 1}\)
E. 0

\[g'(x) = \sqrt{2^x + 1}\]
\[g'(2) = \sqrt{2^2 + 1}\]
\[= \sqrt{5}, \quad A\]
11. [3 marks]
Let \( f(x, y) = x^2y^3 + e^{xy} \). Then \( f_y = 3x^2y^2 + xe^{xy} \)

A. \( 6xy^2 + xye^{xy} \)
B. \( 3y^2 + e^x \)
C. \( 3x^2y^2 + e^{xy} \)
D. \( 3x^2y^2 + ye^{xy} \)
E. \( 3x^2y^2 + xe^{xy} \)

12. [3 marks]
If \( y + xz^2 - x^2z^3 = 2 \) defines \( z \) implicitly as a function of \( x \) and \( y \) near the point \( x = 1, y = 2, z = 1 \), then at that point \( \frac{\partial z}{\partial y} = \)

A. \(-1\)
B. \(1\)
C. \(0\)
D. \(-6\)
E. \(5\)

\[ \frac{\partial z}{\partial y} = \frac{1}{3x^2z^2 - 2xz} = 1 \text{ at } x = 1, y = 2, z = 1, \quad \text{B} \]
13. [3 marks]  
If \( f(x, y) = x^2 e^{y^2} \) then \( f_{xy}(1, 1) = \)  
A. \( 4e \)  
B. \( 12e \)  
C. \( 6e \)  
D. \( 8e \)  
E. \( 2e \)  

\[
f_x = 2xe^{y^2} 
\]
\[
f_{xy} = 4xye^{y^2} = 4e \text{ at } (1, 1), \quad A
\]

14. [3 marks]  
If \( z = x^2 + xy + y^2 \) where \( x = 3t - 6 \) and \( y = t^2 + 2 \) then when \( t = 1 \) \( \frac{dz}{dt} = \)  
A. \( 9 \)  
B. \( 3 \)  
C. \( -3 \)  
D. \( 15 \)  
E. \( 6 \)  

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} 
\]
\[
= (2x + y) * 3 + (x + 2y) * 2t 
\]
when \( t = 1 \) \( x = -3, \ y = 3 \)  
\[
\frac{dz}{dt} = (-6 + 3) * 3 + (-3 + 6) * 2 
\]
\[
= -9 + 6 = -3, \quad C
\]
15. [3 marks]

The joint demand functions for the products $A$ and $B$ are given by:

$$q_A = \frac{200}{p_A \sqrt{p_B}} \quad q_B = \frac{300}{p_B \sqrt[3]{p_A}}$$

Which of the following statements is true?

A. $\frac{\partial q_A}{\partial p_A} > 0$

B. $\frac{\partial q_B}{\partial p_B} > 0$

C. Products $A$ and $B$ are complementary

D. Products $A$ and $B$ are competitive

E. Products $A$ and $B$ are neither complementary nor competitive

$$\frac{\partial q_A}{\partial p_A} = \frac{-200}{p_A \sqrt{p_B}} < 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_B} = \frac{-300}{p_B \sqrt[3]{p_A}} < 0$$

so $A$ and $B$ are false.

$$\frac{\partial q_A}{\partial p_B} = \frac{-100}{p_A (p_B)^{3/2}} < 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} = \frac{-100}{p_B (p_A)^{4/3}} < 0$$

so the goods are complementary. C
PART B. WRITTEN-ANSWER QUESTIONS

B1. [11 marks]
(a) [4 marks]
A 20 year mortgage for $500,000 has monthly payments with interest at 4% compounded semiannually. Find the amount of each payment (to the nearest cent).

If $i$ is monthly rate \((1 + i)^{12} = (1.02)^2\)

\[
500,000 = Ra_{\overline{240} | i}
\]

\[
R = \frac{500,000i}{1 - (1 + i)^{-240}} = \frac{500,000[(1.02)^{1/6} - 1]}{1 - (1.02)^{-40}}
\]

\[R = \$3021.23\]

(b) [3 marks]
Find the principal outstanding (to the nearest cent) in the mortgage of question 1(a), just after the 144th payment has been made.

Principal outstanding is the P.V. of the remaining \(240 - 144 = 96\) payments

\[
P.O. = Ra_{\overline{96} | i} = 500,000a_{\overline{96} | i} = 500,000\left[\frac{1 - (1.02)^{-96}}{1 - (1.02)^{-240}}\right]
\]

\[= 500,000\left[\frac{1 - (1.02)^{-16}}{1 - (1.02)^{-40}}\right] = \$248,171.66\]

(c) [4 marks]
What is the market price (to the nearest cent) of a $100 bond having 9 years until maturity and semiannual coupons, with annual coupon rate 6% and annual yield rate 5%?

\[
P = V(1 + i)^{-n} + rVa_{\overline{n} | r}
\]

\[
V = 100 \quad n = 18 \quad r = .03 \quad i = .025
\]

\[
P = 100(1.025)^{-18} + 3a_{\overline{18} | .025}
\]

\[= \$107.18\]
B2. [11 marks]
Consider the graphs of \( x = y^2 + 1 \) and \( x = 4y + 1 \)
(a) [2 marks]
Find the points where those graphs intersect.

\[
y^2 + 1 = 4y + 1 \\
y^2 - 4y = 0 \\
y(y - 4) = 0 \\
y = 0 \Rightarrow x = 1 \quad y = 4 \Rightarrow x = 17
\]
(1,0) and (17,4) are intersection points

(b) [5 marks]
Express as an integral the finite area bounded by those graphs.

\[
\text{Area} = \int_0^4 [(4y + 1) - (y^2 + 1)] \, dy \\
\text{or}
\]
upper curve is \( y = \sqrt{x-1} \) lower curve is \( y = \frac{x-1}{4} \)
\[
\text{Area} = \int_1^{17} \left [ \sqrt{x-1} - \frac{x-1}{4} \right ] \, dx
\]
(c) [4 marks]
Find the area from part (b).

First way: \( \int_0^4 [(4y + 1) - (y^2 + 1)] \, dy = \left [ 2y^2 - \frac{y^3}{3} \right ]_0^4 = 2 \ast 16 - \frac{64}{3} - 0 = \frac{32}{3} \)

Second way: \( \int_1^{17} \left [ \sqrt{x-1} - \frac{x-1}{4} \right ] \, dx \)
\[
= \left [ \frac{2}{3} (x-1)^{3/2} - \frac{(x-1)^2}{8} \right ]_1^{17} \\
= \frac{2}{3} \ast 16^{3/2} - \frac{1}{8} \ast 16^2 - 0 \\
= \frac{2}{3} \ast 64 - 32 = \frac{32}{3} \text{ as well}
\]
B3. [13 marks]
Evaluate the following integrals

(a) [6 marks]
$$\int_{0}^{1} (2x + 1)e^{2x} \, dx$$

By parts $u = 2x + 1$ $\quad du = 2dx$
$dv = e^{2x} \quad v = \frac{e^{2x}}{2}$

$$\int_{0}^{1} (2x + 1)e^{2x} \, dx = \frac{(2x + 1)e^{2x}}{2}\bigg|_{0}^{1} - \int_{0}^{1} e^{2x} \, dx$$
$$= \frac{3e^{2}}{2} - \frac{1}{2} - \frac{e^{2}}{2}\bigg|_{0}^{1}$$
$$= \frac{3e^{2}}{2} - \frac{1}{2} - (\frac{e^{2}}{2} - \frac{1}{2}) = e^{2}$$

(b) [7 marks]
$$\int_{1}^{\infty} \frac{1}{x^2(x + 1)} \, dx = \lim_{R \to \infty} \int_{1}^{R} \frac{1}{x^2(x + 1)} \, dx$$
$$\frac{1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1} \quad Ax(x + 1) + B(x + 1) + Cx = 1$$
$$x = 0 \Rightarrow B = 1$$
$$x = -1 \Rightarrow C = 1$$
$$x = 1 \Rightarrow 2A + 2B + C = 1$$
$$2A + 2 + 1 = 1 \Rightarrow A = -1$$

$$\lim_{R \to \infty} \int_{1}^{R} \left[ \frac{-1}{x} + \frac{1}{x + 1} + \frac{1}{x^2} \right] \, dx$$
$$= \lim_{R \to \infty} \left[ -\ln |x| + \ln |x + 1| - \frac{1}{x} \right]_{1}^{R}$$
$$= \lim_{R \to \infty} \left[ \ln \frac{R + 1}{R} - (\ln 2 - 1) \right] \quad \frac{R + 1}{R} \to 1 \text{ so } \ln \left( \frac{R + 1}{R} \right) \to 0 \text{ and } \frac{1}{R} \to 0$$
$$= 1 - \ln 2$$
B4. [8 marks]
Assuming that \( y > 0 \), find an expression for \( y \) in terms of \( x \) if \( y \) satisfies the differential equation

\[
\frac{dy}{dx} = xy
\]

and \( y = 3 \) when \( x = 0 \). [Hint: This one’s easy.]

\[
\frac{dy}{y} = xdx
\]

\[
\int \frac{dy}{y} = \int x \, dx
\]

\( y > 0 \)

\[
\ln y = \frac{x^2}{2} + C
\]

\[
y = e^{x^2/2+C} = e^{x^2/2}e^C = Ae^{x^2/2}
\]

when \( x = 0 \) \( \Rightarrow 3 = Ae^0 = A \)

so \( y = 3e^{x^2/2} \)
B5. [12 marks]

(a) [6 marks]
Find and classify the critical point(s) of \( f(x, y) = 5x^2 - 2xy + 2y^2 - 10x + 2y \)

\[
\begin{align*}
    f_x &= 10x - 2y - 10 = 0 & 5x - y &= 5 \\
    f_y &= -2x + 4y + 2 = 0 & -x + 2y &= -1 \\
    & & x - 2y &= 1 \\
    & & 5x - y &= 5 \\
\end{align*}
\]

\[
\begin{pmatrix}
    1 & -2 & 1 \\
    5 & -1 & 5 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
    1 & -2 & 1 \\
    0 & 9 & 0 \\
\end{pmatrix}
\]

\( y = 0 \) and \( x = 1 \)

\[
\begin{align*}
    f_{xx} &= 10 & f_{yy} &= 4 & f_{xy} &= -2 \\
    D &= f_{xx}f_{yy} - (f_{xy})^2 \\
    & = 10 \times 4 - (-2)^2 \\
    & = 36 > 0 \\
(1,0) \text{ is a local extremum} \\
    f_{xx} > 0 \text{ so } (1,0) \text{ is a local min}
\end{align*}
\]

(b) [6 marks]
By using the method of Lagrange multipliers only find the critical points of the joint cost function

\[
c(q_A, q_B) = q_A^2 - q_A q_B + \frac{3}{2} q_B^2 + 300
\]
subject to the constraint

\( q_A + q_B = 700 \).

[Show all your work. No marks will be given for any other method.]

\[
\mathcal{L} = q_A^2 - q_A q_B + \frac{3}{2} q_B^2 + 300 - \lambda(q_A + q_B - 700)
\]

\[
\begin{align*}
    \frac{\partial \mathcal{L}}{\partial q_A} &= 2q_A - q_B - \lambda = 0 & \text{and} & \frac{\partial \mathcal{L}}{\partial \lambda} &= -(q_A + q_B - 700) = 0 \\
    \frac{\partial \mathcal{L}}{\partial q_B} &= -q_A + 3q_B - \lambda = 0 \\
\end{align*}
\]

\( 2q_A - q_B = \lambda \)

\( -q_A + 3q_B = \lambda \)

\( 2q_A - q_B = -q_A + 3q_B \)

\[
\begin{align*}
    3q_A - 4q_B &= 0 \\
    q_A + q_B &= 700
\end{align*}
\]

\[
\begin{pmatrix}
    1 & 1 & 700 \\
    3 & -4 & 0 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
    1 & 1 & 700 \\
    0 & -7 & -2100 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
    1 & 1 & 700 \\
    0 & 1 & 300 \\
\end{pmatrix}
\]

\( q_B = 300 \qquad q_A = 400 \)