Supplementary Questions for HP Chapter 6

1. Solve the following matrix equation for $a$, $b$, $c$ and $d$:

\[
\begin{pmatrix}
  a - b & b + c \\
  3d + c & 2a - 4d
\end{pmatrix} = \begin{pmatrix}
  8 & 1 \\
  7 & 6
\end{pmatrix}
\]

2. Two-Commodity Market Model

Consider a model in which only two commodities are related to each other.

For $i$ equal to 1 or 2, let:

- $Q_{di}$ be the quantity demanded of the commodity $i$
- $Q_{si}$ be the quantity supplied of the commodity $i$
- $P_i$ is the price of commodity $i$.

For simplicity, the demand and supply functions of both commodities are assumed to be linear. Such a model can be written as:

\[
\begin{align*}
Q_{d1} - Q_{s1} &= 0 \\
Q_{d1} &= a_0 + a_1 P_1 + a_2 P_2 \\
Q_{s1} &= b_0 + b_1 P_1 + b_2 P_2 \\
Q_{d2} - Q_{s2} &= 0 \\
Q_{d2} &= \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2 \\
Q_{s2} &= \beta_0 + \beta_1 P_1 + \beta_2 P_2
\end{align*}
\]

where the $a$, $b$, $\alpha$ and $\beta$ are (appropriately) chosen coefficients.

(a) Write the above system as a matrix equation consisting of column matrices, where each column matrix consists of the coefficients that are all associated with the same variable.

(b) Write the above system as a single matrix equation (i.e., in the form $AX = B$, where $A$ is the coefficient matrix and $X$ and $B$ are column matrices).

3. Let $A = \begin{pmatrix}
  6 & 9 & 0 \\
  -4 & -6 & 0 \\
  1 & 3 & 1
\end{pmatrix}$. Find $A^{65}$.

4. (a) Express the equations

\[
\begin{align*}
y_1 &= x_1 - x_2 + x_3 \\
y_2 &= 3x_1 + x_2 - 4x_3 \\
y_3 &= -2x_1 - 2x_2 + 3x_3
\end{align*}
\]

and

\[
\begin{align*}
z_1 &= 4y_1 - y_2 + y_3 \\
z_2 &= -3y_1 + 5y_2 - y_3
\end{align*}
\]
in the matrix forms $Y = AX$ and $Z = BY$. Then use these to obtain a direct relationship
$Z = CX$ between $Z$ and $X$.
(b) Use the equation $Z = CX$ in (a) to express $z_1$ and $z_2$ in terms of $x_1$, $x_2$ and $x_3$.
(c) Check the result in (b) by directly substituting the equations for $y_1$, $y_2$ and $y_3$ into
the equations for $z_1$ and $z_2$ and then simplifying.

5. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$
(a) Find all scalars $k$ such that $AX = kX$ has a non-zero solution $X$.
(b) For each value of $k$ from part (a), find all non-zero $X$ such that $AX = kX$.

6. A restaurant owner plans to use $x$ tables seating four, $y$ tables seating six, and $z$ tables
seating eight, for a total of 20 tables. When fully occupied, the tables seat 108 customers.
If only half of the $x$ tables, half of the $y$ tables and one-fourth of the $z$ tables are used,
each fully occupied, then 46 customers will be seated. Find $x$, $y$, $z$.

7. The pattern of unemployment over a period of time can be described by a Markov
model, the linear system
\[ x_{t+1} = qx_t + py_t \]
\[ y_{t+1} = (1-q)x_t + (1-p)y_t \]
For any $k$, $x_k$ denotes the percentage of people of working age employed at time $k$,
and $y_k$ denotes the percentage of people of working age who are unemployed at time $k$. $p$, $q$
are constants such that $0 \leq p \leq 1$, $0 \leq q \leq 1$.
If the unemployment and employment numbers remain constant, this yields the linear
system
\[ x = qx + py \]
\[ y = (1-q)x + (1-p)y \]
\[ 1 = x + y \]
or
\[ (q-1)x + py = 0 \]
\[ (1-q)x - py = 0 \]
\[ x + y = 1 \]
(a) If $p$ and $q$ lie between 0 and 1, how many solutions does this system have? Why?
(b) Ignoring the condition that $p$ and $q$ lie between 0 and 1, find values of $p$ and $q$ so that
this system has no solutions.

8. A box containing pennies, nickels and dimes has 13 coins with a total value of 83 cents.
How many coins of each type are in the box?
9. Solve the following system by the method of reduction:

\[
\begin{align*}
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 + 15x_6 &= 5 \\
2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6
\end{align*}
\]

\[
\begin{align*}
x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 + 15x_6 &= 5 \\
2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6
\end{align*}
\]

10. Find the values(s) of the constant \( c \) for which the following system

\[
\begin{align*}
x + 3y + 2z &= 0 \\
x + cy + 4z &= 0 \\
2y + cz &= 0
\end{align*}
\]

has infinitely many solutions.

11. Without using row-reduction, determine whether the following matrices are invertible:

(a) \[
A = \begin{bmatrix} 2 & 1 & -3 & 1 \\ 0 & 5 & 4 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}
\]

(b) \[
B = \begin{bmatrix} 5 & 1 & 4 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}
\]

**Hint:** Consider the associated homogeneous system of equations for each matrix. Also, note that if a system \( AX = C \) (including \( C = 0 \)) has a unique solution, then \( A \) is invertible.

12. (a) Let \( A \) be a square matrix. Show that

(i) \( (I - A)^{-1} = I + A + A^2 + A^3 \) if \( A^4 = 0 \)

(ii) \( (I - A)^{-1} = I + A + A^2 + A^3 + \cdots + A^{n-1} \) if \( A^n = 0 \)

(b) Let \( J_n \) be the \( n \times n \) matrix each of whose entries is 1. Show that

\[
(I - J_n)^{-1} = I - \frac{1}{n-1}J_n.
\]

13. Let \( I \) denote the \( n \times n \) identity matrix and let \( A \) be an \( n \times n \) matrix such that \( A^3 - 3A^2 + 2A + I = 0 \). Then \( A^{-1} \)

(a) does not exist

(b) equals \(-A^3 + 3A^2 - 2A\)
(c) equals $A(A - I)(A - 2I)$
(d) equals $(I - A)(A - 2I)$
(e) equals $A^2 - 3A + 2I$

14. A 3-industry sector of an economy is composed of industries $A$, $B$ and $C$. The external demands for their products are $d_A$, $d_B$ and $d_C$ respectively.

(a) Suppose that each dollar of output in any of the industries requires a quarter (25 cents) of output from that industry itself and a quarter (25 cents) from each of the other industries. If external demand is to be satisfied exactly, find the total production of each industry (in terms of $d_A$, $d_B$ and $d_C$). Show the technology matrix and the Leontiff matrix in solving this problem.

(b) As in (a), but each dollar of output of any industry requires $\frac{1}{3}$ of a dollar of input from that industry itself and $\frac{1}{3}$ of a dollar from each of the other industries, give all possible answers and interpret the result.