Supplementary Questions for HP Chapter 10

1. Evaluate \( \lim_{h \to 0} \frac{(x+h)^n-x^n}{h} \), using the identity \( a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \).

2. Show by example that
   (a) \( \lim_{x \to a} [f(x)+g(x)] \) may exist even though neither \( \lim_{x \to a} f(x) \) nor \( \lim_{x \to a} g(x) \) exists.
   (b) \( \lim_{x \to a} [f(x)g(x)] \) may exist even though neither \( \lim_{x \to a} f(x) \) nor \( \lim_{x \to a} g(x) \) exists.

3. Evaluate
   (a) \( \lim_{x \to 0} \frac{\sqrt[3]{1+cx^2} - 1}{x} \)
   (b) \( \lim_{x \to 1} \frac{\sqrt[3]{x-1}}{\sqrt{x+1} - 1} \)

4. A function \( f(x) \) is said to be an even function if \( f(-x) = f(x) \) for all \( x \) in the domain of \( f \), and an odd function if \( f(-x) = -f(x) \) for all \( x \) in the domain of \( f \).

   Suppose \( f(x) \) is even and \( \lim_{x \to a^+} f(x) = L \).
   (a) Find, if possible, \( \lim_{x \to a} f(x) \).
   (b) Find, if possible, \( \lim_{x \to a^-} f(x) \).
   (c) Find, if possible, \( \lim_{x \to a^+} f(x) \).
   (d) Repeat (a) to (c) for an odd function \( f(x) \).

5. Let \( F(x) = \frac{x^2 - 1}{|x-1|} \)
   (a) Find
      i) \( \lim_{x \to 1^+} F(x) \)      ii) \( \lim_{x \to 1^-} F(x) \)
   (b) Does \( \lim_{x \to 1} F(x) \) exist?
   (c) Sketch the graph of \( F \).

6. (a) Let \( r \) be any positive number. Show that \( \frac{1}{r} \ln \left( 1 + \frac{r}{n} \right) \) is the slope of the straight line connecting \( g(1) \) and \( g(1 + \frac{r}{n}) \) for the function \( g(x) = \ln(x) \).
   (b) In light of question (a), what is \( \lim_{n \to \infty} \frac{1}{r} \ln(1 + \frac{r}{n}) \) in terms of the function \( g \)? It may help to write \( h = \frac{r}{n} \).
   (c) Write \( (1 + \frac{r}{n})^n = e^{n \ln(1 + \frac{r}{n})} \). Show \( \lim_{n \to \infty} (1 + \frac{r}{n})^n = e^r \), remembering \( g'(1) = 1 \).

7. (a) In terms of compound interest, explain why it is reasonable to expect that

\[
\left( 1 + \frac{r}{n+1} \right)^{n+1} > \left( 1 + \frac{r}{n} \right)^n
\]
where \( r > 0 \), \( n \) is a positive integer.

(b) Show that \( \left(1 + \frac{r}{n+1}\right)^{n+1} > (1 + \frac{r}{n})^n \) using the identity \( a^b = e^{b \ln a} \) and using the fact that \( \frac{2}{r} \ln \left(1 + \frac{r}{n}\right) \) is the slope of the straight line connecting \( g(1) \) and \( g(1 + \frac{r}{n}) \) for the function \( g(x) = \ln x \).

8. (a) Consider the function \( f(x) = \lim_{y \to -\infty} x^y \), for \( 0 \leq x \leq 1 \). At what point(s) is \( f(x) \) discontinuous?

(b) Consider the function \( f(x) = \lim_{y \to \infty} \frac{x^y}{x^y - 1} \), for \( x \geq 0 \). At what point(s) is \( f(x) \) discontinuous?

9. Consider the function \( f(x) = x^m \) where \( m \) is an integer, with the convention that \( 0^0 = 1 \). What are the condition(s) on \( m \) that indicate whether \( f(x) \) is continuous at \( x = 0 \)?

10. A function \( f(x) \) is said to have a removable discontinuity at \( x = a \) if \( \lim_{x \to a} f(x) \) exists, but either \( f(a) \) is not defined or \( \lim_{x \to a} f(x) \neq f(a) \).

(a) State the (exact) conditions needed for a rational function to have a removable discontinuity at \( x = a \).

(b) Given that the rational function \( \frac{f(x)}{g(x)} \) has a removable discontinuity at \( x = a \), find \( h(x) \) such that:

1) \( h(x) = \frac{f(x)}{g(x)} \) \( (x \neq a) \)

2) \( h(x) \) does not have a removable discontinuity at \( x = a \).

11. Let \( f(x) = \frac{|x^2 - 1|}{x^2 - 1} \).

(a) Explain why \( f(x) \) is continuous wherever it is defined.

(b) For each point where \( f(x) \) is not defined, state whether a value can be assigned to \( f(a) \) in such a way as to make \( f \) continuous at \( a \).