DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

Friday, May 11, 2001

Questions have equal value, but different parts of a question may have different weights.

1. a) Give the definitions for connected space and for path-connected space.
   b) Give an example of a topological space \( X \) that is connected but not path connected.
   c) Give the definitions for Hausdorff space and for normal space.
   d) Prove that every compact Hausdorff space is normal.

2. a) State the Mayer-Vietoris theorem for the homology of a topological space \( X \).
   b) Use the theorem to compute the homology groups of \( S^n \) for all \( n \).
   c) Use the theorem to compute the homology groups of the real projective plane \( \mathbb{R}P(2) \).

3. a) Let \( f : S^n \to S^n \) be a continuous map, \( n \geq 1 \). State the definition of the degree of this map.
   b) What is the degree of the antipodal map
      \[
      a : S^n \to S^n, \quad a(x) = -x,
      \]
      and what is the degree of the identity map? (Give a brief justification of your answer.)
   c) Show that if \( h : S^n \to S^n \) has degree different from that of the antipodal map, then \( h \) has a fixed point.
4. Let $X$ be a topological space, and $R$ a commutative ring.

a) State the formula for the differential $(c_1 \cup c_2)$ of the cup product of two singular cochains $c_1, c_2$ in $X$, with coefficients in $R$.

b) Using a), show that the cup product on cochains induces a ring structure on the cohomology with coefficients in $R$.

c) State the cohomology groups with coefficients in $\mathbb{Z}$ of the 2-torus $X_1 = T^2$ and of the space $X_2 = S^1 \vee S^1 \vee S^2$.

\begin{align*}
\text{Compare the ring structure on the cohomology of } X_1, X_2. \text{ (You are not asked to justify your answer.)}
\end{align*}

5. a) State van Kampen’s theorem.

b) Show that if $X$ is a topological $n$-manifold with $n \geq 3$, the fundamental group $\pi_1(X)$ is unchanged by removal of an $n$-dimensional disk from $X$.

c) Let $X_1, X_2$ be two connected, oriented topological $n$-manifolds, with $n \geq 3$. The connected sum $X_1 \# X_2$ is defined by deleting two disks $D_j \subset X_j$ and identifying the boundaries $\partial(X_j \setminus D_j) \cong S^{n-1}$ by an orientation-reversing homeomorphism. Express the fundamental group of $X_1 \# X_2$ in terms of $\pi_1(X_1), \pi_1(X_2)$.

6. a) Give the definition of a strong deformation retraction of a topological space $X$.

b) Find the first fundamental group of the following spaces. (Give a brief justification of your answer.) Which of these groups are abelian?

(1) $S^2$ with two points removed,
(2) Klein bottle with a point removed,
(3) Torus with two points removed.