DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

January 1997

No aids.
Do all questions.
Each question is worth 20 marks.

1. a) Let \( f: S^2 \times I \to \mathbb{R} \) be continuous. Suppose that \( \min_{p \in S^2} f(p, 0) = 0 \), \( \min_{p \in S^2} f(p, 1) = 1 \).
   Prove that for every \( 0 < m < 1 \), \( \exists 0 < t < 1 \) such that \( \min_{p \in S^2} f(p, t) = m \).
   
b) Does the result in part (a) hold if \( S^2 \) is replaced by \( \mathbb{R}^2 \)? Explain your answer.

2. Let \( X_1 \subset X_2 \subset \cdots \) be a sequence of Hausdorff spaces where \( X_i \) is a closed subspace of \( X_{i+1} \) for each \( i \). Let \( X = \bigcup_{i=1}^{\infty} X_i \). Define the coherent topology on \( X \) by \( U \subset X \) is open \( \iff U \cap X_i \) is open in \( X_i \) \( \forall i \).
   
a) Verify that this is a topology.
   
b) Show that \( X_i \) is a subspace of \( X \) in this topology.
   
c) Suppose that each \( X_i \) is normal; state Tietze’s extension theorem and use it to show that \( X \) is normal.

3. Let \( M \) be a compact connected manifold. (Recall that a manifold is Hausdorff and locally Euclidean, that is, every point has a neighbourhood homeomorphic to an open set in \( \mathbb{R}^n \) for some \( n \).) Let \( \pi: P \to M \) be the universal cover. Show that \( P \) is compact \( \iff \pi_1(M) \) is finite.
4. Let $X$ be the outline of the tetrahedron; that is, $X = \bigcup_{i=1}^{6} L_i \bigcup_{i=1}^{4} P_i$

where $L_i$ are the edges and $P_i$ are the vertices. Calculate $H_*(X; \mathbb{Z})$.

5. a) Calculate $\pi_1(S^2 \times S^1)$.
   
   b) Calculate $\pi_1(S^2 \times \Delta)$ where $\Delta = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.
   
   c) Notice that $S^2 \times S^1 \cong S^2 \times \partial \Delta$. Show that $S^2 \times S^1$ is not a retract of $S^2 \times \Delta$. 

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