1. (a) Define the “quotient topology”.

(b) Consider the system of differential equations:

\[
\begin{align*}
\dot{x} &= x, \\
\dot{y} &= -y,
\end{align*}
\]

\((x, y) \in \mathbb{R}^2\).

The general solution is \(x = ae^t, \ y = be^{-t}\).

The points on any orbit (solution curve) satisfy \(xy = c\) for some \(c\). The orbits are drawn below.
There are 9 possible types of orbits, given explicitly as follows:

for $c > 0$

$T_1(c) = \{(x, y) \in \mathbb{R}^2 \mid xy = c$ and $x > 0, \ y > 0\}$

$T_2(c) = \{(x, y) \in \mathbb{R}^2 \mid xy = c$ and $x < 0, \ y < 0\}$

for $c < 0$

$T_3(c) = \{(x, y) \in \mathbb{R}^2 \mid xy = c$ and $x > 0, \ y < 0\}$

$T_4(c) = \{(x, y) \in \mathbb{R}^2 \mid xy = c$ and $x < 0, \ y > 0\}$

for $c = 0$

$T_5 = \{(x, 0) \in \mathbb{R}^2 \mid x > 0\}$

$T_6 = \{(0, y) \in \mathbb{R}^2 \mid y > 0\}$

$T_7 = \{(x, 0) \in \mathbb{R}^2 \mid x < 0\}$

$T_8 = \{(0, y) \in \mathbb{R}^2 \mid y < 0\}$

$T_9 = \{0, 0\}$.

The orbit space $\mathcal{M}$ is defined as $\mathcal{M} = \mathbb{R}^2 \sim$ with the quotient topology where $(x, y) \sim (x', y')$ if they lie on the same orbit.

Describe $\mathcal{M}$ as a topological space; that is, give a basis for the open sets. Is $\mathcal{M}$ Hausdorff? Why or why not?

2. Let $X$ be a path connected space. Suppose that there exists a continuous map $m: X \times X \to X$ and a point $e \in X$ such that $m(e, x) = m(x, e) = x$ for all $x \in X$. Show that the fundamental group $\pi_1(X, e)$ is abelian.

3. (a) Define what it means for a topological space to be “normal”.

(b) State Urysohn’s Lemma.

(c) Prove that a compact Hausdorff space is normal.

4. Let $sq: S^1 \to S^1$ be the map $z \mapsto z^2$ where the circle is regarded as the unit ball of $\mathbb{C}$ and the multiplication is that from $\mathbb{C}$. Compute the group homomorphism $sq_*: \pi_1(S^1) \to \pi_1(S^1)$.

5. Let $M$ be an $n$-dimensional path connected topological manifold (so that each point has a neighbourhood which is homeomorphic to $\mathbb{R}^n$), and let $x \in M$. Compute the relative homology $H_q(M, M - x)$ for all $q$.

6. Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a short exact sequence of chain complexes. Define the connecting homomorphism $\Delta : H_n(C) \to H_{n-1}(A)$. Show that the sequence $H_n(B) \xrightarrow{j} H_n(C) \xrightarrow{\Delta} H_{n-1}(A)$ is exact at $H_n(C)$.