1. Let $X$ be connected and let $C$ be a closed subset such that the boundary of $C$ is a single point. Show that $C$ is connected.

2. Let $X$ be the comb space defined by

$$X = \left( \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) \mid 0 \leq y \leq 1\} \right) \cup \{(0, y) \mid 0 \leq y \leq 1\} \cup \{(x, 0) \mid 0 \leq x \leq 1\} \subset \mathbb{R}^2.$$ 

Let $I = \{(0, y) \mid 0 \leq y \leq 1\} \subset X$.

(i) Sketch $X$.

(ii) Define deformation retract and strong deformation retract.

(iii) Show that $I$ is a deformation retract of $X$.

(iv) Show that $I$ is not a strong deformation retract of $X$.

3. (i) Suppose $n \geq 2$. Does there exist a continuous map $f : S^n \to S^1$ which is not homotopic to a constant?

(ii) Suppose $n \geq 2$. Does there exist a continuous map $f : \mathbb{R}P^n \to S^1$ which is not homotopic to a constant?

(iii) Let $T = S^1 \times S^1$ be the torus. Does there exist a continuous map $f : T \to S^1$ which is not homotopic to a constant?

In each case, carefully justify your answer.
4. Let $p : E \to X$ be a covering map and let $f : Y \to X$ be any continuous map. Let $P = \{(y, e) \in Y \times E \mid fy = pe\}$ and define $\pi : P \to Y$ by $\pi(y, e) = y$. Show that $\pi$ is a covering map.

5. Let $f : S^{n-1} \to Y$ be continuous ($n > 1$) and let $Y_f = D^n \cup_f Y$. Show that:

(i) $H_m(Y) \cong H_m(Y_f)$ for $m \neq n, n - 1$.

(ii) There is an exact sequence

$$0 \to H_n(Y) \to H_n(Y_f) \to H_{n-1}(S^{n-1}) \to H_{n-1}(Y) \to H_{n-1}(Y_f) \to 0.$$

6. Compute $H_*(S^n)$. 

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