Real Analysis Exam (2 hours)

May 1999

No aids.
Do all questions.
Questions will be weighted equally.

1. (a) What is the dual of $L^3(\mathbb{R})$?

(b) Show that the dual of $\ell^\infty$ ("bounded functions on the positive integers") is not $\ell^1$ ("summable functions on the positive integers"), by exhibiting an element of the dual that is not in $\ell^1$.

2. Let $f$ be a non-negative integrable function on $\mathbb{R}$ (with Lebesgue measure), let $\mu$ be Lebesgue measure on $\mathbb{R}^2$, and show that

$$
\mu\left(\{(x,y) : 0 \leq y \leq f(x)\}\right) = \mu\left(\{(x,y) : 0 < y < f(x)\}\right) = \int f(x) \, dx.
$$

3. For some measures, $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, $L^r \supset L^s$, and for some measures, $L^r$ can never contain $L^s$ unless $r = s$. Find and explain examples of each phenomenon and/or necessary and/or sufficient conditions.

4. Let $\{\delta_n\}$ be a sequence of positive numbers, and $\{\phi_n\}$ an orthonormal set in an infinite-dimensional Hilbert space $\mathcal{H}$. Set

$$
S = \{x = \sum_{n=1}^{\infty} a_n \phi_n \in \mathcal{H} : |a_n| \leq \delta_n\}.
$$

Prove $S$ is compact if and only if $\sum \delta_n^2 < \infty$. (In the case $\delta_n = \frac{1}{n}$, $S$ is called the "Hilbert cube").
5. Find the maximum value of $\int_{-1}^{1} x^3 g(x) \, dx$, for measurable functions $g(x)$ satisfying

$$\int_{-1}^{1} g(x) \, dx = \int_{-1}^{1} xg(x) \, dx = \int_{-1}^{1} x^2 g(x) \, dx = 0,$$

and $\int_{-1}^{1} |g(x)|^2 \, dx = 1$.

6. Evaluate the derivative and the second derivative of the Heaviside function $H$ on $\mathbb{R}$, in the sense of distributions; the Heaviside function is:

$$H(x) = \begin{cases} 
0, & \text{if } x < 0 \\
1, & \text{if } x \geq 0 
\end{cases}$$