DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam (2 hours)

May 1, 2000

No aids.
Do all questions.
The value of each is indicated. Total = 80

1. (a) [6 marks]
Evaluate \( \int_\gamma \frac{dz}{z^2-1} \) where \( \gamma \) is the circle \(|z-ia|=1\) traversed once counterclockwise and \( a > 1 \).

(b) [6 marks]
Evaluate \( \int_\gamma \frac{z e^{z}}{(z-b)^2} dz \) where \( \gamma \) is the unit circle traversed once counterclockwise and \(|b| \neq 1\).

(c) [8 marks]
Let \( U \) be the open unit disc, i.e. \( U = \{ z \in \mathbb{C} \mid |z| < 1 \} \). Is \( U \setminus \{0\} \) biholomorphic to \( \mathbb{C} \setminus \{0\} \)? Explain.

2. [20 marks]
A family \( \mathcal{F} \) of holomorphic functions on a domain \( \Omega \) is said to be normal if any sequence of functions in \( \mathcal{F} \) contains a subsequence which converges uniformly on compact subsets of \( \Omega \). Suppose that the family \( \mathcal{F} \) fails to be normal. Show that there exists a point \( z_0 \in \Omega \) such that \( \mathcal{F} \) is not normal in any neighbourhood of \( z_0 \).

Hint: a compactness argument.

3. [20 marks]
Suppose that \( f \) and \( g \) are meromorphic in \( \mathbb{C} \). Suppose that \( \gamma \) is a smooth closed curve in \( \mathbb{C} \) (not necessarily simple) such that neither \( f \) nor \( g \) has any singularities on \( \gamma \). Suppose also that \(|g| < |f|\) on \( \gamma \). Let
\[
\varphi(t) = \int_\gamma \frac{f'(z) + t g'(z)}{f(z) + t g(z)} \, dz \quad 0 \leq t \leq 1.
\]

Show that \( \varphi \) is constant.

4. [20 marks]
Evaluate \( \int_0^\infty \frac{x^\alpha}{x^2 + a^2} \, dx \) via residues where \(-1 < \alpha < 1 \) and \( a > 0 \).