DEPARTMENT OF MATHEMATICS  
University of Toronto

Complex Analysis Exam (2 hours)

September 7, 1999

No aids.
Do all questions.
Questions will be weighted equally.

1. Suppose that $f$ is analytic in $|z| < R_0$ and that $|a| < R < R_0$. Evaluate

$$
\int_\gamma \frac{(R^2 - |a|^2)f(z)}{(z - a)(R^2 - z\bar{a})} \, dz
$$

where $\gamma$ is the circle $|z| = R$ transversed counterclockwise. Hence prove that if $0 < r < R$

$$
f(re^{i\theta}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\phi})d\phi}{R^2 - 2rR\cos(\theta - \phi) + r^2}
$$

(Poisson’s formula).

2. (a) Show that if $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z^n} + \frac{1}{n} \right)$ then $f$ is analytic in the whole plane minus the points $0, 1, 2, \ldots$.

(b) Find $f^{(k)}(z)$. (Justify.)

3. Find the image of the upper half plane under the mapping $w = \frac{1-z^n}{1+z^n}$ where $0 < \alpha < 1$ and $z^\alpha$ has its principal value.

4. Suppose $f$ is entire and $|f(z)| \leq C_1 + C_2|z|$. What can you say about the form of $f$? Give a proof.