DEPARTMENT OF MATHEMATICS  
University of Toronto

Practice exam in Analysis (3 hours)

1. How many roots does the equation $z^8 - 2z^5 + 6z^3 - z + 1 = 0$ have in the region $|z| < 1$?

2. (i) Find one 1–1 onto conformal map $f$ that sends the open quadrant $\{(x, y) : x > 0$ and $y > 0\}$ onto the open lower half disc $\{(x, y) : x^2 + y^2 < 1$ and $y < 0\}$.
   (ii) Find all such $f$.

3. (i) Define almost everywhere convergence and convergence in $L_1$-norm.
   (ii) Show by example that neither form of convergence implies the other.
   (iii) Prove that any sequence which is Cauchy in $L_1$-norm has a subsequence which converges a.e.

4. (i) Define the space $S$ of Schwartz functions on $\mathbb{R}$.
   (ii) State the Fourier inversion theorem.
   (iii) Prove that the Fourier transform $f \mapsto \hat{f}$ maps $S$ onto $S$.

5. (i) Define the spectrum of a bounded linear operator $T$ on a Hilbert space $\mathcal{H}$.
   (ii) What is meant by compactness of such a $T$?
   (iii) If $\mathcal{H}$ has an orthonormal basis $\{e_n\}_{n=1}^\infty$ and $\{a_n\}$ is a sequence of complex numbers converging to 0 define $T$ by $Te_n = a_ne_n$. Prove directly that $T$ is compact. What is the spectrum of $T$?

6. (i) What is a tempered distribution on $\mathbb{R}$?
   (ii) Define the derivative of a tempered distribution.
   (iii) Show that
   $$\langle F, f \rangle = \int_{-\infty}^{\infty} \log |x| f(x) dx$$
   defines a tempered distribution $F$ and that
   $$\langle F', f \rangle = PV \int_{-\infty}^{\infty} \frac{1}{x} f(x) dx \equiv \lim_{\epsilon \rightarrow 0^+} \int_{\{x > \epsilon\}} \frac{1}{x} f(x) dx.$$